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# Power corrections to $\bar{B} \to X_u \ell \bar{\nu}$ ($X_s \gamma$) decay spectra in the "shape-function" region

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Abstract: Using soft-collinear effective theory (SCET), we examine the $1/m_b$ corrections to the factorization formulas for inclusive semi-leptonic $B$ decays in the endpoint region, where the hadronic final state consists of a single jet. At tree level, we find a new contribution from four-quark operators that was previously assumed absent. Beyond tree level many sub-leading shape-functions are needed to correctly describe the decay process.

Keywords: Heavy Quarks Physics, B-Physics, QCD.



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## Contents



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## 1. Introduction

Inclusive semi-leptonic and radiative $B$ meson decays offer many opportunities to test the flavour sector of the Standard Model. The radiative decay $\bar{B} \to X_s\gamma$ can be used as a probe for new physics, and the most precise measurements of the CKM matrix element $|V_{ub}|$ are based on the semi-leptonic decay $\bar{B} \to X_u\ell\bar{\nu}$. However, experimental cuts on the final state are needed to suppress a strong background from charm production. These cuts often force the kinematics of the decay into the so-called shape-function region, where the hadronic final state is collimated into a single jet, which carries a large energy of order $m_b$, but a small invariant mass of order $m_b\Lambda_{\rm QCD}$. While the conventional operator product expansion (OPE) breaks down in this region of phase-space, it is still possible to make rigorous predictions using a twist expansion, which sums singular terms in the OPE into non-local operators evaluated on the light-cone [1 − 3]. This method amounts to using QCD factorization formulas which separate the physics from the disparate mass scales $m_b^2 \gg m_b\Lambda_{\rm QCD} \gg \Lambda_{\rm QCD}^2$ into hard, jet-, and shape-functions, respectively. To the above mentioned decays these ideas have been applied in [4, 5]. The recent development of the soft-collinear effective theory (SCET) as applied to inclusive decays [6 − 9] has provided a natural framework from which to prove these QCD factorization formulas to all orders in perturbation theory, and also to sum the Sudakov logarithms appearing therein; such studies have been performed to leading order in $1/m_b$ [7, 10, 11].

The goal of the present work is to use SCET to analyze the structure of the factorization formulas beyond leading order in $1/m_b$. The motivation for this is two-fold. First, from the phenomenological side, the inclusion of power-suppressed effects will soon be necessary to keep theoretical predictions on par with the improving experimental accuracy being achieved at the $B$ factories. Second, from a purely theoretical standpoint, although the machinery needed to begin a discussion of power corrections to QCD factorization formulas within SCET has been available for some time, there have been no theoretical efforts which apply this to a concrete example. For inclusive decays a study of sub-leading shape-function effects was carried out previously in [12 − 15] by more elementary methods, and restricted to the tree approximation. In this paper, we take the first steps towards analyzing power-suppressed effects in inclusive $B$ decays in the shape-function region beyond this approximation by an investigation of the general structure of factorization, and an enumeration of all relevant shape-functions. We perform a complete calculation of the hadronic tensor only at tree level. However, even in this approximation we find a new effect related to certain four-quark operators, and some discrepancies with earlier results.

The factorization formulas are derived by performing a two-step matching from QCD → SCET → HQET, identifying the hard functions in the first step of matching, and the jet- and shape-functions in the second. The first matching is done at scales of order $m_b$, the second at scales $\sqrt{m_b\Lambda_{\rm QCD}}$. Since the hard and jet-functions can therefore be computed perturbatively in the strong coupling, it is evident that factorization extends to sub-leading order in $1/m_b$. The advantage of the effective theory framework is that it provides a transparent book-keeping of all the relevant interactions at every step in the calculation, including power-suppressed effects, and allows us to identify all the HQET operators that define the sub-leading shape-functions with no restriction to tree level in the hard ($\mu \sim m_b$) and hard-collinear ($\mu \sim \sqrt{m_b\Lambda_{\rm QCD}}$) fluctuations.

The content of the paper is as follows: sections 2 to 4 provide an outline of the structure of factorization at sub-leading order in $1/m_b$, including a basis for the SCET currents and HQET shape-functions needed at order $1/m_b$, but to any order in the strong coupling $\alpha_s$. In section 5 we pick one of the many terms in the expansion and demonstrate its reduction to a convolution of a jet- and a shape-function. In section 6 we compute the structure functions that parameterize the semi-leptonic and radiative decay spectra in the tree approximation at order $1/m_b$. We find many structural simplifications at tree level, but also a tetra-local term that has been missed before. Sections 7 and 8 follow up on these observations by commenting on the nature of possible simplifications beyond tree level and by providing a numerical estimate of the correction from the new tetra-local contribution. We conclude in section 9.

## 2. Factorization at sub-leading order

The effects of QCD in semi-leptonic $B$ decays are contained in the hadronic tensor $W^{\mu\nu}$. We can calculate the hadronic tensor via the optical theorem by taking the imaginary part of the forward scattering amplitude, which we define as

$$W^{\mu\nu} = \frac{1}{\pi}\,\mathrm{Im}\langle \bar{B}(v)|T^{\mu\nu}|\bar{B}(v)\rangle\,. \tag{2.1}$$

We use the state normalization $\langle \bar{B}(v)|\bar{B}(v)\rangle$=1 and drop the velocity label from now on. The correlator $T^{\mu\nu}$ is the time-ordered product of two flavour-changing weak currents

$$T^{\mu\nu} = i\int d^4x e^{-iq\cdot x}\mathrm{T}\{J^{\dagger\mu}(x), J^\nu(0)\}\,, \tag{2.2}$$

where $q$ is the momentum carried by the outgoing leptonic pair ($\bar{B} \to X_u\ell\bar{\nu}$) or photon ($\bar{B} \to X_s\gamma$). The calculation of this correlator is the essential element to obtaining the decay amplitudes. Any differential decay distribution can be derived from the components of the hadronic tensor. We shall use the notation of [16] and write the hadronic tensor in terms of five scalar structure functions

$$\begin{aligned} W_{\mu\nu} = {} & W_1\,(p_\mu v_\nu + v_\mu p_\nu - g_{\mu\nu}vp - i\epsilon_{\mu\nu\alpha\beta}\,p^\alpha v^\beta) - \\ & -W_2\,g_{\mu\nu} + W_3\,v_\mu v_\nu + W_4\,(p_\mu v_\nu + v_\mu p_\nu) + W_5\,p_\mu p_\nu\,, \end{aligned} \tag{2.3}$$

where the independent vectors are chosen to be $v$, the velocity of the $\bar{B}$ meson, and $p \equiv m_bv - q$ with $m_b$ the $b$ quark pole mass. (We use the convention $\epsilon^{0123} = -1$.) The $W_i$ are regarded as functions of $p^2$ and $vp$. The relation of these partonic variables to the final state hadronic invariant mass $P^2$ and energy $vP$ is

$$vp = vP - (M_B - m_b)\,, \qquad p^2 = P^2 - 2(M_B - m_b)vP + (M_B - m_b)^2\,. \tag{2.4}$$

Of course, there is a dependence of the $W_i$ on the current $J$ assumed in (2.2), which is not made explicit in our notation. Since we are interested in either the semi-leptonic or radiative decay, the relevant currents are[1]

$$J^\mu = \bar{q}\gamma^\mu(1-\gamma_5)b\,, \qquad J^\mu = \frac{1}{2}\bar{q}\,[\gamma^\mu, \not{q}](1+\gamma_5)b\,. \tag{2.5}$$

[1]For radiative decays the restriction to currents implies that we consider only the contribution from the leading electromagnetic penguin operator in the effective weak hamiltonian. The complete result contains additional terms from four-quark operators.

The decay kinematics are assumed to be in the so-called shape-function or SCET region, where the hadronic jet emitted at the weak vertex carries a large energy of order $m_b$, but a small invariant mass squared of order $m_b\Lambda_{\rm QCD}$. Such a jet is referred to as hard-collinear, and implies the existence of three widely separated mass scales $m_b^2 \gg m_b\Lambda_{\rm QCD} \gg \Lambda_{\rm QCD}^2$, along with a small parameter $\lambda^2 = \Lambda_{\rm QCD}/m_b$. The soft-collinear effective theory (SCET) offers a tool with which to calculate the decay amplitudes as a series in this small parameter $\lambda$. An arbitrary momentum $p$ is decomposed as

$$p^\mu = n_+p\,\frac{n_-^\mu}{2} + p_\perp^\mu + n_-p\frac{n_+^\mu}{2}\,, \tag{2.6}$$

where $n_\pm^\mu$ are light-like vectors satisfying $n_+n_- = 2$. For inclusive decays it is sufficient to use what is referred to in the literature as $\mathrm{SCET_I}$, which contains only hard-collinear and soft degrees of freedom. The components of a hard-collinear momentum are defined to scale as $(n_+p_{hc},\, p_{hc\perp},\, n_-p_{hc}) \sim m_b(1,\, \lambda,\, \lambda^2)$ and those of a soft momentum as $p_s \sim m_b(\lambda^2,\, \lambda^2,\, \lambda^2)$. From now on we will refer to hard-collinear momenta as simply collinear, and to $\mathrm{SCET_I}$ as SCET. We find it convenient to work in a frame of reference where $v_\perp = 0$ and $n_-v = n_+v = 1$. Furthermore, the SCET expansion in $\lambda$ refers to a frame in which the total transverse momentum of the hadronic final state is at most of order $\lambda$. For any given $q$, we therefore choose the frame where $q_\perp = [m_bv - p]_\perp = -p_\perp = 0$ and compute the invariant components of the hadronic tensor in this frame.

It has been shown that the decay amplitude factorizes into a convolution of hard, jet, and soft functions at leading order in $\lambda$ [4, 7]. Using SCET, our aim is to show that an analogous factorization holds at sub-leading order in $\lambda$. The scaling properties of the collinear jet imply that no Lorentz invariant quantity can be formed at order $\lambda$, so the leading power corrections appear at order $\lambda^2$. In technical terms, transverse momenta of order $\lambda$ appear only in the internal integrations over the collinear momenta in a jet, and since the integral over an odd number of transverse momenta either vanishes or must be proportional to one of the external soft transverse momenta of order $\lambda^2$, the resultant expansion is in powers of $\lambda^2 \sim 1/m_b$. We therefore need to go to second order in the SCET expansion. Before giving any explicit formulae, we will outline a procedure that as a matter of principle could be used to establish factorization at any order in $\lambda$.

### 2.1 Matching to SCET/HQET

In the first step we remove the hard scale $m_b^2$ as a dynamical scale by matching the QCD lagrangian and currents onto their corresponding expressions in HQET and SCET, where fluctuations are characterized by the jet scale $m_b\Lambda_{\rm QCD}$. We then use these effective theory quantities to calculate the correlator in (2.2). In the following we use the position space formulation of SCET [8, 9], which is especially well suited for the study of power corrections, since the power-suppressed lagrangians and currents are already known.

We first discuss the SCET/HQET lagrangian. Its explicit form to order $\lambda^2$ is [9]

$$\mathcal{L} = \bar\xi\left(in_-D + i\not{D}_{\perp c}\frac{1}{in_+D_c}i\not{D}_{\perp c}\right)\frac{\not{n}_+}{2}\xi - \frac{1}{2}\,\mathrm{tr}\left(F_c^{\mu\nu}F^c_{\mu\nu}\right) +$$

$$
+\bar{h}_v ivD_s h_v + \bar{q}_s i\not{D}_s q_s - \frac{1}{2}\,\mathrm{tr}\left(F_s^{\mu\nu}F^s_{\mu\nu}\right) +
$$
$$
+\mathcal{L}_\xi^{(1)} + \mathcal{L}_{\xi q}^{(1)} + \mathcal{L}_{\mathrm{YM}}^{(1)} + \sum_{i=1}^{3}\mathcal{L}_{\xi i}^{(2)} + \mathcal{L}_{\xi q}^{(2)} + \mathcal{L}_{\mathrm{YM}}^{(2)} + \mathcal{L}_{\mathrm{HQET}}^{(2)}. \tag{2.7}
$$

The $\xi$ denotes the collinear quark field, $q_s$ the soft quark field, and $h_v$ the heavy quark field of HQET. The covariant derivatives are defined as $iD_c = i\partial + gA_c$ and analogously for $iD_s$, but a $D$ without subscript contains both the collinear and soft gluon field. The quantities $F^s_{\mu\nu}$ ($F^c_{\mu\nu}$) are the field-strength tensors built from the soft (collinear) gauge fields in the usual way, except for the definition of $F^c_{\mu\nu}$, where $n_-D$ rather than $n_-D_c$ appears. The collinear and soft fields are evaluated at $x$, but in products of soft and collinear fields the soft fields are evaluated at $x_-^\mu = (n_+x/2)\,n_- \equiv x_+n_-^\mu$, according to the multipole expansion. In this notation $x_+$ is a scalar, while $x_-^\mu$ is a vector.

The power-suppressed terms in the effective lagrangian read

$$
\begin{aligned}
\mathcal{L}_\xi^{(1)} &= \bar{\xi}x_\perp^\mu n_-^\nu W_c\, gF^s_{\mu\nu}W_c^\dagger\frac{\not{n}_+}{2}\xi\,,\\
\mathcal{L}_{1\xi}^{(2)} &= \frac{1}{2}\bar{\xi}n_-xn_+^\mu n_-^\nu W_c\, gF^s_{\mu\nu}W_c^\dagger\frac{\not{n}_+}{2}\xi\,,\\
\mathcal{L}_{2\xi}^{(2)} &= \frac{1}{2}\bar{\xi}x_\perp^\mu x_{\perp\rho}n_-^\nu W_c[D^\rho_{\perp s}, gF^s_{\mu\nu}]W_c^\dagger\frac{\not{n}_+}{2}\xi\,,\\
\mathcal{L}_{3\xi}^{(2)} &= \frac{1}{2}\bar{\xi}i\not{D}_{\perp c}\frac{1}{in_+D_c}x_\perp^\mu\gamma_\perp^\nu W_c\, gF^s_{\mu\nu}W_c^\dagger\frac{\not{n}_+}{2}\xi +\\
&\quad+\frac{1}{2}\bar{\xi}x_\perp^\mu\gamma_\perp^\nu W_c\, gF^s_{\mu\nu}W_c^\dagger\frac{1}{in_+D_c}i\not{D}_{\perp c}\frac{\not{n}_+}{2}\xi\,,\\
\mathcal{L}_{\xi q}^{(1)} &= \bar{q}_sW_c^\dagger i\not{D}_{\perp c}\xi - \bar{\xi}i\overleftarrow{\not{D}}_{\perp c}W_cq_s\,,
\end{aligned} \tag{2.8}
$$

where the $W_c$ are collinear Wilson lines. We have omitted the terms $\mathcal{L}_{\xi q}^{(2)}$, since their field content implies that they do not contribute to the current correlator at order $\lambda^2$. The explicit expressions of the Yang-Mills lagrangians $\mathcal{L}_{\mathrm{YM}}^{(1,2)}$ can be found in [9]. They are needed only in the calculation of $1/m_b$ corrections beyond tree level. The SCET lagrangian is exact to all orders in perturbation theory, receiving no radiative corrections [8]. In contrast, the $\lambda^2$ HQET lagrangian is

$$
\mathcal{L}_{\mathrm{HQET}}^{(2)} = \frac{1}{2m_b}\left[\bar{h}_v(iD_s)^2h_v + \frac{C_{\mathrm{mag}}}{2}\,\bar{h}_v\sigma_{\mu\nu}gF_s^{\mu\nu}h_v\right], \tag{2.9}
$$

where $C_{\mathrm{mag}} \neq 1$ represents the renormalization of the chromomagnetic interaction by hard quantum fluctuations.

A second source of hard corrections is related to the matching onto the SCET heavy-light currents. The QCD currents $J_i = \bar{\psi}\Gamma_iQ$ are represented in SCET as convolutions of dimensionless short-distance Wilson coefficients depending on quantities at the hard scale $m_b^2$ with current operators $J_j^{(k)}$ composed of HQET and SCET fields. The matrix elements of these effective currents are characterized by fluctuations on the order of the jet scale $m_b\Lambda_{\mathrm{QCD}}$ and the soft scale $\Lambda^2_{\mathrm{QCD}}$. We write this convolution as

$$
(\bar{\psi}\Gamma_iQ)(x) = e^{-im_bv\cdot x}\sum_{j,k}\widetilde{C}_{ij}^{(k)}(\hat{s}_1,\ldots,\hat{s}_n)\otimes J_j^{(k)}(\hat{s}_1,\ldots,\hat{s}_n;x)\,, \tag{2.10}
$$

where the $\otimes$ stands for a convolution over a set of dimensionless variables $\hat{s}_i \equiv s_i m_b$. The superscript $k$ refers to the scaling of the current operator with $\lambda$ relative to the leading-power currents, and the subscript $j$ enumerates the effective currents at a given order in $\lambda$. As with the SCET lagrangian, the collinear fields in the current operator $J_j^{(k)}$ depend on $x + s_i n_+$ but the soft fields including $h_v$ are multipole-expanded and depend only on $x_-$. The factorization formula will eventually be formulated in terms of convolutions over longitudinal collinear momentum fractions, which we define as $u_i = n_+ p_i/m_b$. The momentum space coefficient functions are related to those defined above by

$$C_{ij}^{(k)}(u_i) = \int \prod_i d\hat{s}_i \, \widetilde{C}_{ij}^{(k)}(\hat{s}_i) \, e^{i\sum_i \hat{s}_i u_i} \,. \tag{2.11}$$

A basis for the most general set of order $\lambda^2$ currents including radiative corrections has not yet been discussed in the literature. We shall return to this point in section 3. For now, we simply note that these short-distance Wilson coefficients depend only on quantities at the hard scale $m_b^2$ and are identified with the hard functions in the factorization formula.

We have now achieved the factorization of the hard scale from the soft and collinear degrees of freedom and can write the correlator as

$$T^{\mu\nu} = \tilde{H}_{jj'}(\hat{s}_1, \ldots, \hat{s}_n) \otimes T_{jj'}^{\mathrm{eff},\mu\nu}(\hat{s}_1, \ldots, \hat{s}_n) \,, \tag{2.12}$$

The hard function $\tilde{H}_{jj'}$ is a product of SCET Wilson coefficients $\tilde{C}_{ij}^{(k)} \tilde{C}_{i'j'}^{(k')}$ (along with a possible $C_{\mathrm{mag}}$ from the HQET lagrangian) and $T_{jj'}^{\mathrm{eff},\mu\nu}$ is a correlator of SCET currents. In what follows we will suppress all indices and denote the correlator by $T^{\mathrm{eff}}$.

### 2.2 Collinear-soft factorization

In the second step we factorize the matrix element of $T^{\mathrm{eff}}$ into soft and collinear pieces. Towards this end, we first redefine the collinear fields according to [7]

$$\xi = Y\xi^{(0)} \,, \qquad A_c = Y A_c^{(0)} Y^\dagger \,, \qquad W_c = Y W_c^{(0)} Y^\dagger \,, \tag{2.13}$$

and immediately drop the superscript on the redefined fields. The $Y$ above is a soft Wilson line involving $n_- A_s$ evaluated at $x_-$, and the redefinition of the collinear Wilson line follows from that of the gluon field. The effect of this redefinition on the SCET lagrangian and currents is to transform every $n_- D$ into $n_- D_c$, and to replace $q_s$ ($h_v$) by $Y^\dagger q_s$ ($Y^\dagger h_v$) and $iD_\mu^s$ ($F_{\mu\nu}^s$) by $Y^\dagger iD_\mu^s Y$ ($Y^\dagger F_{\mu\nu}^s Y$). Note that the collinear Wilson line appears only in $W_c^\dagger \xi$ and $W_c^\dagger iD_\mu^c W_c$, so the positions of soft and collinear Wilson lines in any field product can always be inferred from the transformation of the fields under the collinear and soft gauge symmetries.

We shall treat the power-suppressed lagrangian terms in the interaction picture. Since after the collinear field redefinition the leading order SCET lagrangian (see (2.7)) no longer couples the collinear to the soft fields, we can factorize the matrix element. The $\bar{B}$ meson by definition contains no collinear degrees of freedom, meaning that the $\bar{B}$ meson state is represented as the tensor product $|\bar{B}\rangle \otimes |0\rangle$, where the first (second) factor refers to the soft (collinear) Hilbert space. It follows that the matrix element of any SCET current correlator, including in general time-ordered products with sub-leading interactions from the lagrangian, can be written as

$$\langle \bar{B}|T^{\rm eff}(\hat{s}_1,\ldots,\hat{s}_n)|\bar{B}\rangle = i\int d^4x d^4y\ldots e^{i(m_b v-q)x}\langle \bar{B}|\bar{h}_v[\text{soft fields}]h_v|\bar{B}\rangle(x_-,y_-,\ldots)\times \\ \times\langle 0|[\text{collinear fields}]|0\rangle(\hat{s}_1,\ldots,\hat{s}_n;x,y,\ldots)\,. \tag{2.14}$$

The additional integrals over $d^4y\ldots$ are related to insertions of the power suppressed lagrangian. The soft matrix element depends only on $n_+x, n_+y,\ldots$, so the integrations over transverse positions and the $n_-$ components can be lumped into the definition of the collinear factor. The soft and collinear matrix elements are then linked by multiple convolutions over the light-cone variables $x_+, y_+,\ldots$.

### 2.3 Definition of "shape-functions" and "jet-functions"

The matrix element of the soft fields between $\bar{B}$ meson states defines non-perturbative (leading and sub-leading) shape-functions $\tilde{S}(n_+x,\ldots)$. We define these in momentum space according to

$$\tilde{S}(x_{1+},\ldots,x_{n+}) = \int d\omega_1\ldots d\omega_n\, e^{-i(\omega_1 x_{1+}+\ldots+\omega_n x_{n+})}\, S(\omega_1,\ldots,\omega_n)\,. \tag{2.15}$$

We shall see below that at order $\lambda^2$ we can have up to a triple convolution over the variables $\omega_i$, coming from two insertions of the order $\lambda$ SCET lagrangian. These scalar functions need to be modeled and introduce hadronic uncertainties into any phenomenological applications. Properly identifying the independent functions is therefore an important task to which we devote sections 4 and 7.

The matrix elements of the collinear fields between the collinear vacuum involve fluctuations with virtualities $m_b\Lambda_{\rm QCD}$. After integration over position arguments they define perturbatively calculable jet-functions, which have the form

$$\mathcal{J}(u_1,\ldots,u_i;\omega_1,\ldots\omega_n)\,. \tag{2.16}$$

(Alternatively, we shall use the variables $p^2_{\omega_{12\ldots j}}$, where $p_{\omega_{12\ldots j}} = p-\omega_{12\ldots j}\,n_+/2$ and $\omega_{12\ldots j} = \omega_1+\omega_2+\ldots\omega_j$.) Evaluating these functions removes the collinear degrees of freedom and defines the final step of matching, SCET $\rightarrow$ HQET. The complicated form of the power-suppressed SCET currents and lagrangian imply that many new jet-functions appear at order $\lambda^2$ and we will not list the general set in this paper. Instead, we simply note that they are perturbative objects and so introduce no inherent theoretical uncertainties.

Having carried out these steps, we can express the correlator in terms of a factorization formula. We insert the definitions of the hard coefficients, jet-functions, and shape-functions and work in momentum space. A generic term in the factorization formula is a sum of convolutions over hard, jet, and shape-functions

$$T = \sum H(u_1,\ldots u_i)\otimes\mathcal{J}(u_1,\ldots,u_i;\omega_1,\ldots,\omega_n)\otimes S(\omega_1,\ldots,\omega_n)\,. \tag{2.17}$$

To discuss more precisely the structure of this factorization formula, we need to identify the set of jet- and shape-functions appearing at order $\lambda^2$. For the jet-functions, the necessary step is to derive the full set of heavy-light currents in the presence of radiative corrections. We turn to this topic in the next section.

### 2.4 Caveats to factorization

The operators whose matrix elements define the jet- and shape-functions have singularities related to the light-cone expansion. Above it has been assumed that there exists a regularization and subtraction procedure that is compatible with the properties of the lagrangian crucial to factorization. To the best of our knowledge, dimensional regularization is adequate for this purpose, but we do not have a general proof of this statement.

The factorization formula (2.17) is composed of convolutions over the soft light-cone variables $\omega_i$, and we must assume that the convolutions of the perturbative jet-functions with the shape-functions converge. Little is known about the functional dependence of sub-leading shape-functions, but a divergence of the convolution for $\omega_i \to 0$ would be surprising, since it would indicate that long-distance physics is not accounted for by the HQET lagrangian. On the other hand, divergences for $\omega_i \to \infty$, if they existed, would be of short-distance nature and could presumably be treated with a modification of the factorization procedure.

## 3. SCET currents

In this section we discuss the matching of the QCD heavy-light currents onto SCET. The order $\lambda$ operators have been investigated in several places [8, 9, 17] including one-loop radiative corrections to their coefficients [18–20], but the order $\lambda^2$ case has been given only at tree level [8, 9]. To give a general factorization formula for the correlator, we will need a complete basis for the heavy-light currents in the presence of radiative corrections.

The QCD currents $J_i = \bar{\psi}\Gamma_i Q$ are represented in SCET as a convolution of dimensionless short-distance Wilson coefficients with a current operator $J_j^{(k)}$ composed of heavy-quark and SCET fields, as in (2.10). Our task is to find a set of current operators $J_j^{(k)}$ up to order $\lambda^2$ relative to the leading-order currents. We work in the frame where $v_\perp = 0$, which considerably reduces the number of allowed currents at a given order in $\lambda$.

In currents at position $x$, all soft fields are evaluated at $x_- = (n_+x/2)n_-$, but the collinear fields may be shifted along the light-cone to positions $x + s_i n_+$ owing to the non-locality of SCET. The variables $s_i$ are integrated in a convolution product of coefficient functions and current operators, which allows us to eliminate factors of $in_+\partial$ and $1/(in_+\partial)$ from the operator by a redefinition of the coefficient function. In the following we use the convention that operators do not contain $in_+\partial$ (or its inverse) operating on collinear fields, and denote the position argument $x + s_i n_+$ of collinear field products by a subscript "$s_i$". It is also convenient to write down the operator basis independent of the Dirac structure $\Gamma_i$ of the QCD current. In this case we assume that the short-distance coefficients are Lorentz-tensors $C_{ij\ldots}$, where the dots stand for further indices on the effective current. We can always decompose these into scalar functions using $n_-^\mu$, $v^\mu$, $g^{\mu\nu}$, and $i\epsilon^{\mu\nu\rho\sigma}$. The scalar functions depend of course on the Dirac structure of the QCD current.

### 3.1 Leading order

At leading order the only possible SCET currents are

$$J^{(0)} = (\bar{\xi} W_c)_s \Gamma_j h_v \,. \tag{3.1}$$

Here $\Gamma_j = \{1, \gamma_5, \gamma_{\alpha_\perp}\}$ denotes a basis of the four independent Dirac matrices between $\bar{\xi}$ and $h_v$. For a given QCD current, we would proceed to decompose the tensor coefficient functions into scalar ones. For instance, for the V-A current relevant to semi-leptonic decay the three tensor coefficient functions $C_\mu, C_{5\mu}$, and $C_{\mu\alpha_\perp}$ multiplying the $J_j^{(0)}$ may be expressed in terms of three scalar coefficient functions multiplying the current operators

$$(\bar{\xi} W_c)_s (1+\gamma_5) \left\{ \gamma^\mu, v^\mu, \frac{n_-^\mu}{n_- v} \right\} h_v \,. \tag{3.2}$$

### 3.2 Next-to-leading order

At relative order $\lambda$ the basis consists of the currents

$$\begin{aligned}
J_1^{(1)} &= (\bar{\xi} W_c)_s \Gamma_j (x_\perp D_s h_v) \\
J_2^{(1)} &= (\bar{\xi} W_c)_s \, i \overleftarrow{\partial}_\perp^\mu \Gamma_j h_v \\
J_3^{(1)} &= (\bar{\xi} W_c)_{s_1} [W_c^\dagger i D_{\perp c}^\mu W_c]_{s_2} \Gamma_j h_v \,.
\end{aligned} \tag{3.3}$$

The above currents should be multiplied by an overall factor $1/m_b$ to make the coefficient functions dimensionless. The basis of operators with scalar coefficient functions can be found in [17 − 19]. Here we use a basis where the transverse derivative is taken outside the collinear Wilson line. This has the advantage that only the third operator has a tree level matrix element with a transverse collinear gluon.

The first two operators are "two-body", that is, they depend only on $x_-$ and one other position $x + sn_+$. The coefficient functions of any two-body operator to any order in the $\lambda$ expansion can be related to the coefficient function of the leading-power currents. The reason for this is that they all follow from matching the QCD matrix element $\langle q|J_i|b\rangle$ to SCET, and the number of independent form factors needed to parameterize the QCD matrix element is equal to the number of independent currents at leading power [18]. The coefficients of the two-body operators follow from the expansion of the full QCD form factors. The operator $J_3^{(1)}$ is a three-body operator, whose coefficient is determined by the expansion of $\langle qg|J_i|b\rangle$ with a transverse gluon to leading power.

### 3.3 Next-to-next-to-leading order

Many structures are possible at this order. We find

$$J_1^{(2)} = \frac{1}{2} (\bar{\xi} W_c)_s \Gamma_j (n_- x n_+ D_s h_v)$$



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$$
\begin{aligned}
J_2^{(2)} &= \frac{1}{2}\,(\bar\xi W_c)_s\Gamma_j(x_{\perp\mu}x_{\perp\nu}D_s^\mu D_s^\nu h_v)\\
J_3^{(2)} &= (\bar\xi W_c)_s\, i\overleftarrow{\partial}_\perp^\mu\Gamma_j(x_\perp D_s h_v)\\
J_4^{(2)} &= (\bar\xi W_c)_s\Gamma_j(iD_s^\mu h_v)\\
J_5^{(2)} &= (\bar\xi W_c)_s\, in_-\overleftarrow{D}_s\Gamma_j h_v\\
J_6^{(2)} &= (\bar\xi W_c)_s\, i\overleftarrow{\partial}_\perp^\mu i\overleftarrow{\partial}_\perp^\nu\Gamma_j h_v\\
J_7^{(2)} &= (\bar\xi W_c)_{s_1}[W_c^\dagger iD_{\perp c}^\mu W_c]_{s_2}\Gamma_j(x_\perp D_s h_v)\\
J_8^{(2)} &= (\bar\xi W_c)_{s_1}\, i\overleftarrow{\partial}_\perp^\mu[W_c^\dagger iD_{\perp c}^\nu W_c]_{s_2}\Gamma_j h_v\\
J_9^{(2)} &= (\bar\xi W_c)_{s_1}\Big[i\,\partial_\perp^\mu(W_c^\dagger iD_{\perp c}^\nu W_c)_{s_2}\Big]\Gamma_j h_v\\
J_{10}^{(2)} &= (\bar\xi W_c)_{s_1}[W_c^\dagger in_- D W_c]_{s_2}\Gamma_j h_v\\
J_{11}^{(2)} &= (\bar\xi W_c)_{s_1}\Big[[W_c^\dagger iD_{\perp c}^\mu W_c]_{s_2},[W_c^\dagger iD_{\perp c}^\nu W_c]_{s_3}\Big]\Gamma_j h_v\\
J_{12}^{(2)} &= (\bar\xi W_c)_{s_1}\Big\{[W_c^\dagger iD_{\perp c}^\mu W_c]_{s_2},[W_c^\dagger iD_{\perp c}^\nu W_c]_{s_3}\Big\}\Gamma_j h_v\\
J_{13}^{(2)} &= (\bar\xi W_c)_{s_1}\mathrm{tr}\Big[[W_c^\dagger iD_{\perp c}^\mu W_c]_{s_2}[W_c^\dagger iD_{\perp c}^\nu W_c]_{s_3}\Big]\Gamma_j h_v\\
J_{14}^{(2)} &= \Big[(\bar\xi W_c)_{s_1}\Gamma_j h_v\Big]\Big[(\bar\xi W_c)_{s_2}\frac{\not n_+}{2}\,\Gamma_{j'}(W_c^\dagger\xi)_{s_3}\Big]\\
J_{15}^{(2)} &= \Big[(\bar\xi W_c)_{s_1}\Gamma_j T^A h_v\Big]\Big[(\bar\xi W_c)_{s_2}\frac{\not n_+}{2}\,\Gamma_{j'}T^A(W_c^\dagger\xi)_{s_3}\Big]\,.
\end{aligned}
\tag{3.4}
$$

The trace refers to colour, and an overall factor $1/m_b^2$ should be included to make the coefficient functions dimensionless. The argument for the completeness of this basis is as follows. The QCD matrix element $\langle q|J_i|b\rangle$ is expanded to order $\lambda^2$, which involves the transverse momentum squared, $n_-$ times the collinear momentum, or the heavy quark residual momentum. Together with the terms from the multipole expansion of the heavy quark field, this accounts for the two-body operators $J_{1-6}^{(2)}$. Next, we can have $\langle qg|J_i|b\rangle$ expanded to first order in the transverse momentum of the collinear quark or the collinear gluon, which gives $J_{7-9}^{(2)}$, with $J_7^{(2)}$ coming from the multipole expansion. The coefficient of $J_7^{(2)}$ is related to that of the order $\lambda$ three-body operator $J_3^{(1)}$. Some of the scalar coefficients of $J_{8,9}^{(2)}$ are also related to $J_3^{(1)}$. However, contrary to the case of two-body operators it is no longer true that the coefficients of all sub-leading three-body operators are related to the leading one, because the number of invariant form factors needed to parameterize $\langle qg|J_i|b\rangle$ in QCD is larger than the number of the leading three-body currents. We also get new three-body operators from $\langle qg|J_i|b\rangle$ with a gluon corresponding to a $n_-A_c$ field or $n_-A_s$, but gauge invariance requires that the soft gluon is part of a covariant derivative, so we get the three-body operator $J_{10}^{(2)}$. In fact, $J_5^{(2)}$ and $J_{10}^{(2)}$ could be eliminated using the collinear quark and gluon equations of motion. Finally, we get four-body operators from $\langle qgg|J_i|b\rangle$ with two transverse collinear gluons or from $\langle qq\bar q|J_i|b\rangle$ with three collinear (anti-)quarks. There are several possible colour structures, which can be chosen as in $J_{11-15}^{(2)}$.

It is useful to gain some intuition as to how this general set of heavy-light currents translates into terms in the factorization formula. First, it is obvious that insertions of heavy-light currents which contain no soft gluon fields simply build up a set of power-suppressed jet-functions convoluted with the leading order shape-function. This includes two insertions of $J^{(1)}_{2,3}$ or single insertions of $J^{(2)}_{6,8-15}$. If we work at tree level, only two-body operators can contribute to the current correlator, and we need only consider two insertions of $J^{(1)}_{1,2}$ or a single insertion of $J^{(2)}_{1-6}$. The full set of three- and four-body operators is only needed when one aims at an accuracy $\alpha_s\Lambda_{\rm QCD}/m_b$ in the calculation of the hadronic tensor.

## 4. Basis of shape-functions

In this section we will collect the results for the independent matrix elements of soft fields ("shape-functions") needed to parameterize the hadronic tensor at order $\lambda^2$. The possible time-ordered products that build up the current correlator to this order are

$$\begin{aligned}
&a)\;\; J^{(0)}J^{(2)}_k + \text{sym.}\\
&b)\;\; J^{(1)}_k J^{(1)}_l\\
&c)\;\; J^{(0)}J^{(1)}_k\mathcal{L}^{(1)} + \text{sym.}\\
&d)\;\; J^{(0)}J^{(0)}\mathcal{L}^{(2)}\\
&e)\;\; J^{(0)}J^{(0)}\mathcal{L}^{(1)}\mathcal{L}^{(1)}\,.
\end{aligned}\tag{4.1}$$

Inspection of the effective currents and lagrangian shows that from these products we obtain the following soft operators (leaving out colour and spinor indices):

1. From a) and b)

$$\begin{aligned}
&(\bar h_v Y)(Y^\dagger h_v)\,,\;(\bar h_v Y)(Y^\dagger iD^\mu_s h_v)\,,\;(\bar h_v(-i)\overleftarrow{D}^\mu_s Y)(Y^\dagger h_v)\,,\\
&(\bar h_v(-i)\overleftarrow{D}^{\mu_\perp}_s(-i)\overleftarrow{D}^{\nu_\perp}_s Y)(Y^\dagger h_v)\,.
\end{aligned}\tag{4.2}$$

   We leave out operators related by hermitian conjugation. The convention is such that fields in different parentheses stand at different positions $0, x_-, z_-, \ldots$ and that colour indices in brackets are contracted. Hence all these terms are bi-local.

2. From the single lagrangian insertions c) and d) in addition

$$\begin{aligned}
&(\bar h_v Y)(Y^\dagger ign^\nu_- F^s_{\mu_\perp\nu}Y)(Y^\dagger h_v)\,,\;(\bar h_v(-i)\overleftarrow{D}^{\rho_\perp}_s Y)(Y^\dagger ign^\nu_- F^s_{\mu_\perp\nu}Y)(Y^\dagger h_v)\,,\\
&(\bar h_v Y)(Y^\dagger ign^\mu_+ n^\nu_- F^s_{\mu\nu}Y)(Y^\dagger h_v)\,,\;(\bar h_v Y)(Y^\dagger igF^s_{\mu_\perp\nu_\perp}Y)(Y^\dagger h_v)\,,\\
&(\bar h_v Y)(Y^\dagger[iD^{\rho_\perp}_s, ign^\nu_- F^s_{\mu_\perp\nu}]Y)(Y^\dagger h_v)\,,\;(\bar h_v Y)(Y^\dagger[in_-D_s, igF^s_{\mu_\perp\nu_\perp}]Y)(Y^\dagger h_v)\,,\\
&(\bar h_v Y)(Y^\dagger h_v)\, i\int d^4z\,\mathcal{L}^{(2)}_{\rm HQET}(z)\,.
\end{aligned}\tag{4.3}$$

   These are tri-local (with exception of the last line). The second-to-last operator comes only from the Yang-Mills lagrangian $\mathcal{L}^{(2)}_{\rm YM}$.

3. From the double insertions e)

$$\begin{aligned}
&(\bar h_v Y)(Y^\dagger ign^\nu_- F^s_{\mu_\perp\nu}Y)(Y^\dagger ign^\sigma_- F^s_{\rho_\perp\sigma}Y)(Y^\dagger h_v)\,,\\
&(\bar h_v Y)(Y^\dagger h_v)(\bar q_s Y)(Y^\dagger q_s)\,.
\end{aligned}\tag{4.4}$$

   These are tetra-local.

We note immediately the increase in complexity of the matrix elements needed to parameterize the $1/m_b$ corrections to all orders in perturbation theory, since there appear tetra-local light-cone operators including a four-quark operator which has not yet been discussed in the literature.

It is convenient to parameterize the shape-functions in a covariant derivative basis. Reinstating colour and spinor indices, the field products listed above can be derived from

$$\begin{aligned}
&(\bar{h}_v Y)(x_-)_{a\alpha}(Y^\dagger h_v)(0)_{b\beta}\,,\\
&(\bar{h}_v Y)(x_-)_{a\alpha}(Y^\dagger iD_s^\mu Y)(z_-)_{cd}(Y^\dagger h_v)(0)_{b\beta}\,,\\
&(\bar{h}_v Y)(x_-)_{a\alpha}(Y^\dagger iD_{s\perp}^\mu Y)(z_{1-})_{cd}(Y^\dagger iD_{s\perp}^\nu Y)(z_{2-})_{ef}(Y^\dagger h_v)(0)_{b\beta}\,,
\end{aligned} \tag{4.5}$$

and the expressions in the last lines of (4.3), (4.4), respectively, using in particular the identities $Y^\dagger in_- D_s Y = in_-\partial$,

$$Y^\dagger ig n_-^\nu F^s_{\mu\nu} Y = -\Big[in_-\partial, Y^\dagger iD^s_\mu Y\Big] = -\Big[in_-\partial Y^\dagger [iD^s_\mu Y]\Big]\,. \tag{4.6}$$

The derivatives are meant to act on everything to the right irrespective of position argument, but derivatives in square brackets (not to be confused with commutators) as in the last expression act only within the square brackets. With this convention

$$(Y^\dagger iD_s^\mu Y)(z)_{cd} = i\partial^\mu \delta_{cd} + (Y^\dagger [iD_s^\mu Y])(z)_{cd}\,. \tag{4.7}$$

Note that since in the second term the derivative acts only on $Y$, this term is a colour octet (in light-cone gauge, $n_- A_s = 0$, and we have $Y^\dagger[iD_s^\mu Y] = gA_s^\mu$). The first term is a colour-singlet, but does not depend on $z$. Hence if, for instance, $(Y^\dagger iD_s^\mu Y)(z)_{cd}$ appears in a tri-local matrix element, the colour-singlet part is only bi-local.

We decompose the matrix elements of (4.5) into a set of scalar shape-functions. The only possible Dirac structures between two static quark fields $\bar{h}_v \ldots h_v$ are 1 and $(\gamma^\mu - \not{v} v^\mu)\gamma_5$. Defining $\epsilon^\perp_{\mu\nu} \equiv i\epsilon_{\mu\nu\rho\sigma} n_-^\rho v^\sigma$, we write (recall that $x_-^\mu = x_+ n_-^\mu$)

$$\langle \bar{B}|(\bar{h}_v Y)(x_-)_{a\alpha}(Y^\dagger h_v)(0)_{b\beta}|\bar{B}\rangle = \frac{\delta_{ba}}{N_c}\frac{1}{2}\left(\frac{1+\not{v}}{2}\right)_{\beta\alpha}\tilde{S}(x_+)\,, \tag{4.8}$$

$$\begin{aligned}
&\langle \bar{B}|(\bar{h}_v Y)(x_-)_{a\alpha}(Y^\dagger h_v)(0)_{b\beta}\, i\int d^4z\,\mathcal{L}^{(2)}_{\rm HQET}(z)|\bar{B}\rangle =\\
&\quad= \frac{1}{2m_b}\frac{\delta_{ba}}{N_c}\frac{1}{2}\left(\frac{1+\not{v}}{2}\right)_{\beta\alpha}\left[\tilde{s}_{\rm kin}(x_+) + C_{\rm mag}\left(\frac{m_b}{\mu}\right)\tilde{s}_{\rm mag}(x_+)\right],
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
&\langle \bar{B}|(\bar{h}_v Y)(x_-)_{a\alpha}(Y^\dagger iD_s^\mu Y)(z_-)_{cd}(Y^\dagger h_v)(0)_{b\beta}|\bar{B}\rangle =\\
&\quad= \frac{1}{2}\left(\frac{1+\not{v}}{2}\right)_{\beta\alpha}\left\{\frac{\delta_{ba}\delta_{cd}}{N_c}\left[-i\tilde{S}'(x_+)v^\mu + \Big(i\tilde{S}'(x_+) - \tilde{T}_1(x_+,0)\Big)n_-^\mu\right] + \right.\\
&\qquad\left. + \frac{2\,T^A_{ba}T^A_{cd}}{N_c^2-1}\left[\tilde{T}_1(x_+,z_+)n_-^\mu\right]\right\} + \frac{1}{2}\left(\frac{1+\not{v}}{2}\gamma_{\rho\perp}\gamma_5\frac{1+\not{v}}{2}\right)_{\beta\alpha}\times\\
&\quad\times\frac{\epsilon_\perp^{\mu\rho}}{2}\left\{\frac{\delta_{ba}\delta_{cd}}{N_c}\Big(\tilde{t}(x_+) - \tilde{T}_2(x_+,0)\Big) + \frac{2\,T^A_{ba}T^A_{cd}}{N_c^2-1}\tilde{T}_2(x_+,z_+)\right\},
\end{aligned} \tag{4.10}$$

$$
\begin{aligned}
&\langle\bar B|(\bar h_v Y)(x_-)_{a\alpha}(Y^\dagger iD^s_{\mu_\perp}Y)(z_{1-})_{cd}(Y^\dagger iD^s_{\nu_\perp}Y)(z_{2-})_{ef}(Y^\dagger h_v)(0)_{b\beta}|\bar B\rangle =\\
&\quad=\frac{1}{2}\left(\frac{1+\not v}{2}\right)_{\beta\alpha}\frac{g^\perp_{\mu\nu}}{2}\Bigg\{\frac{\delta_{ba}\delta_{cd}\delta_{ef}}{N_c}\,\tilde u_1(x_+)+\frac{2\,\delta_{cd}T^A_{ba}T^A_{ef}}{N_c^2-1}\frac{1}{2}\Big[\tilde U_1(x_+,z_{2+})-\tilde U_2(x_+,z_{2+})-\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad-\tilde{\mathcal U}_1(x_+,z_{2+},z_{2+})\Big]+\\
&\qquad\qquad\qquad\qquad\qquad+\frac{2\,\delta_{ef}T^A_{ba}T^A_{cd}}{N_c^2-1}\frac{1}{2}\Big[\tilde U_1(x_+,z_{1+})+\tilde U_2(x_+,z_{1+})-\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad-\tilde{\mathcal U}_1(x_+,z_{1+},z_{1+})\Big]+\\
&\qquad\qquad\qquad\qquad\qquad+\frac{2\,\delta_{ba}T^A_{cd}T^A_{ef}}{N_c^2-1}\Big[\tilde{\mathcal U}_1(x_+,z_{1+},z_{2+})-\tilde{\mathcal U}_2(x_+,z_{1+},z_{2+})-\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad-\tilde{\mathcal U}_3(x_+,z_{1+},z_{2+})\Big]\\
&\qquad\qquad\qquad\qquad\qquad-\frac{4if^{ABC}T^A_{cd}T^B_{ef}T^C_{ba}}{N_c(N_c^2-1)}\tilde{\mathcal U}_2(x_+,z_{1+},z_{2+})+\\
&\qquad\qquad\qquad\qquad\qquad+\frac{4N_c d^{ABC}T^A_{cd}T^B_{ef}T^C_{ba}}{(N_c^2-1)(N_c^2-4)}\tilde{\mathcal U}_3(x_+,z_{1+},z_{2+})\Bigg\}+\\
&\qquad+\left(-\frac{1}{2}\right)\left(\frac{1+\not v}{2}\not n_-\gamma_5\frac{1+\not v}{2}\right)_{\beta\alpha}\frac{\epsilon^\perp_{\mu\nu}}{2}\Big\{\tilde u_1\to 0,\ \tilde U_{1,2}\to\tilde U_{3,4},\tilde{\mathcal U}_{1,2,3}\to\tilde{\mathcal U}_{4,5,6}\Big\},
\end{aligned}
\tag{4.11}
$$

$$
\begin{aligned}
&\langle\bar B|(\bar h_v Y)(x_-)_{a\alpha}(Y^\dagger h_v)(0)_{b\beta}\,(\bar q_s Y)(z_{1-})_{c\gamma}(Y^\dagger q_s)(z_{2-})_{d\delta}|\bar B\rangle =\\
&\quad=\frac{\delta_{da}\delta_{bc}}{N_c^2}\Bigg\{\frac{1}{2}\left(\frac{1+\not v}{2}\right)_{\beta\alpha}\Big[\delta_{\delta\gamma}\,\tilde{\mathcal V}_1(x_+,z_{1+},z_{2+})+\not v_{\delta\gamma}\,\tilde{\mathcal V}_2(x_+,z_{1+},z_{2+})+\\
&\qquad\qquad\qquad+\not n_{-\delta\gamma}\,\tilde{\mathcal V}_3(x_+,z_{1+},z_{2+})+(\not n_-\not v)_{\delta\gamma}\,\tilde{\mathcal V}_4(x_+,z_{1+},z_{2+})\Big]+\\
&\qquad\quad+\frac{1}{2}\left(\frac{1+\not v}{2}\gamma^\mu\gamma_5\frac{1+\not v}{2}\right)_{\beta\alpha}\Big[(\gamma_\mu\gamma_5)_{\delta\gamma}\,\tilde{\mathcal V}_5(x_+,z_{1+},z_{2+})+\\
&\qquad\qquad\qquad\qquad+(\not v\gamma_\mu\gamma_5)_{\delta\gamma}\,\tilde{\mathcal V}_6(x_+,z_{1+},z_{2+})+\\
&\qquad\qquad\qquad\qquad+(\not n_-\gamma_\mu\gamma_5)_{\delta\gamma}\,\tilde{\mathcal V}_7(x_+,z_{1+},z_{2+})+\\
&\qquad\qquad\qquad\qquad+(\not n_-\not v\gamma_\mu\gamma_5)_{\delta\gamma}\,\tilde{\mathcal V}_8(x_+,z_{1+},z_{2+})\Big]\Bigg\}+\\
&\qquad+\frac{4\,T^A_{da}T^A_{bc}}{(N_c^2-1)}\Bigg\{\frac{1}{2}\left(\frac{1+\not v}{2}\right)_{\beta\alpha}\Big[\tilde{\mathcal V}_{1-4}\to\tilde{\mathcal V}_{9-12}\Big]+\\
&\qquad\qquad+\frac{1}{2}\left(\frac{1+\not v}{2}\gamma^\mu\gamma_5\frac{1+\not v}{2}\right)_{\beta\alpha}\Big[\tilde{\mathcal V}_{5-8}\to\tilde{\mathcal V}_{13-16}\Big]\Bigg\}.
\end{aligned}
\tag{4.12}
$$

This decomposition makes no assumption on whether the external state is a $\bar B$ meson, a $\bar B^*$ meson or a $b$-hadron, but it assumes that it is averaged over polarizations, so that the only available vectors are $v$ and $n_-$. Our notation uses lowercase letters for shape-functions depending on a single variable, capital letters for those depending on two variables, and calligraphic letters for those depending on three variables. The only exception is our notation for the leading order shape-function $\tilde{S}(x_+)$, where we use a capital letter even though it depends only on a single variable. The multi-locality of a given shape-function is determined by decomposing the colour structure of the covariant derivatives $(Y^\dagger iD_s^\mu Y)(z)_{cd}$ as in (4.7), and then using that the singlet component has a lower degree of non-locality than the octet component. The parameterization is chosen such that the colour contractions with the tree-level jet-functions, as well as the limits $z=0$ in (4.10) and $z_1=z_2$ in (4.11), take a simple form. For instance, writing the combination $\tilde{t}(x_+)-\tilde{T}_2(x_+,0)$ for the colour singlet term in the fourth line of (4.10) ensures that only $\tilde{t}(x_+)$ appears at tree-level. We used the heavy quark equation of motion to reduce the number of independent functions. The gluon field equation can be used to eliminate $(Y^\dagger in_+D_sY)(z)_{cd}$. This does not lead to a simplification in practice, and we have therefore kept $\tilde{T}_1(x_+,z_+)$ as a basis shape-function. $\tilde{S}'$ is the derivative of $\tilde{S}$ with respect to $x_+$.

It follows that to order $1/m_b$, but to arbitrary order in $\alpha_s$ in the coefficient functions, the differential decay rates depend on a large number of multi-local shape-functions. Fortunately, the structure of the result is much simpler in the tree approximation as we shall see below. In addition to displaying the general set of shape-functions, the above enumeration of soft matrix elements allows us to clarify in more detail the structure of the convolutions in the factorization formula (2.17). Since the effective currents contain at most three products of collinear fields at different positions, the number of collinear convolutions is at most two. The shape-functions are at most tetra-local, so the maximal number of soft convolution integrals at order $1/m_b$ is three. This results in the structure

$$\begin{aligned} T = {} & H\cdot \mathcal{J}(\omega)\otimes S(\omega)+\sum H(u_1,u_2)\otimes \mathcal{J}(u_1,u_2;\omega)\otimes S(\omega)+ \\ & +\sum H(u)\otimes \mathcal{J}(u;\omega_1,\omega_2)\otimes S(\omega_1,\omega_2)+\sum H\cdot \mathcal{J}(\omega_1,\omega_2,\omega_3)\otimes S(\omega_1,\omega_2,\omega_3)+\cdots\,, \end{aligned} \tag{4.13}$$

where the ellipses denote $1/m_b^2$ terms not considered here, and for each term the most complicated structure is shown. The momentum space shape-functions are defined in terms of the coordinate space shape-functions given above by the Fourier transform (2.15). The variable $\omega_i$ corresponds to $n_-k_i$, where $k_i$ is the (outgoing) momentum of soft fields.

## 5. Example of factorization at order $1/m_b$

We now have all the ingredients needed to calculate the factorization formula to order $\lambda^2\sim 1/m_b$. To illustrate the step-by-step procedure outlined in section 2, we work out the contribution from the time-ordered product

$$T\Big\{J^{(0)\dagger}(x)J_2^{(1)}(0)\,i\int d^4z\,\mathcal{L}_\xi^{(1)}(z)\Big\} \tag{5.1}$$

as an example. All other terms can be treated in a similar way. However, there is a large number of them, and we do not list them in this paper explicitly.

Specifically, we consider the currents

$$J^{(0)}=(\bar\xi W_c)_{s_1}\gamma^\mu h_v\,,\qquad J_2^{(1)}=(\bar\xi W_c)_{s_1}i\overleftarrow{\not\partial}_\perp\gamma^\mu h_v\,, \tag{5.2}$$

which appear in the SCET expansion of the vector current. In the first step we obtain the corresponding contribution to the current correlation function $T^{\mu\nu}$ $(p = m_b v - q)$,

$$\begin{aligned} T^{\mu\nu} &= i\int d^4x\, e^{ip\cdot x}\int d\hat{s}_1 d\hat{s}_2\, \tilde{C}^{(0)\star}(\hat{s}_1)\tilde{C}_2^{(1)}(\hat{s}_2)\,\times \\ &\quad\times T\Bigg\{\bar{h}_v(x_-)\gamma^\mu(W_c^\dagger\xi)(x+s_1n_+)\Big[(\bar{\xi}W_c)(s_2n_+)i\overleftarrow{\partial\!\!\!/}_\perp\Big]\gamma^\nu\, h_v(0)\,\times \\ &\qquad\times i\int d^4z\,(\bar{\xi}W_c)(z)z_\perp^\lambda n_-^\rho gF^s_{\lambda\rho}(z_-)\frac{n\!\!\!/_+}{2}(W_c^\dagger\xi)(z)\Bigg\}. \end{aligned} \tag{5.3}$$

The coefficient function $\tilde{C}_2^{(1)}$ can be related to $\tilde{C}^{(0)}$ as explained in section 3, but the detailed form of this relation is not important for the following. We now use translation invariance to shift all fields by the amount $-s_2n_+$, then perform the change of variables $x \to x + (s_2 - s_1)n_+$, $z \to z + s_2n_+$. This does not affect the position of the soft fields, since $[s_in_+]_- = 0$. The integrations over $\hat{s}_{1,2}$ can then be performed to obtain

$$\begin{aligned} T^{\mu\nu} &= H\left(\frac{n_+p}{m_b}\right)i\int d^4x\, e^{ip\cdot x}\, T\Bigg\{\bar{h}_v(x_-)\gamma^\mu(W_c^\dagger\xi)(x)\Big[(\bar{\xi}W_c)(0)i\overleftarrow{\partial\!\!\!/}_\perp\Big]\gamma^\nu\, h_v(0)\,\times \\ &\qquad\times i\int d^4z(\bar{\xi}W_c)(z)z_\perp^\lambda n_-^\rho gF^s_{\lambda\rho}(z_-)\frac{n\!\!\!/_+}{2}(W_c^\dagger\xi)(z)\Bigg\} \end{aligned} \tag{5.4}$$

with $H(n_+p/m_b) = C^{(0)}(n_+p/m_b)C_2^{(1)}(n_+p/m_b)$. ($C^{(0)}$ is real.) The effects of the hard scale have turned into a multiplicative factor $H$ rather than a convolution, because both effective currents were only two-body. Next we perform the collinear field redefinition that decouples collinear and soft fields. We also use (4.6) and integrate by parts to let the $in_-\partial_{(z)}$ act on the collinear fields. The result is

$$\begin{aligned} \langle\bar{B}|T^{\mu\nu}|\bar{B}\rangle &= H\left(\frac{n_+p}{m_b}\right)i\int d^4xd^4z\, e^{ip\cdot x}\, z^{\rho_\perp}\,\times \\ &\quad\times\langle\bar{B}|(\bar{h}_vY)(x_-)_{a\alpha}(Y^\dagger[iD^s_{\rho_\perp}Y])(z_-)_{cd}(Y^\dagger h_v)(0)_{b\beta}|\bar{B}\rangle\,\times \\ &\quad\times in_-\partial_{(z)}\langle 0|\,T\Big\{(\gamma^\mu W_c^\dagger\xi)(x)_{a\alpha}\Big[(\bar{\xi}W_c)i\overleftarrow{\partial\!\!\!/}_\perp\gamma^\nu\Big](0)_{b\beta}(\bar{\xi}W_c)(z)_c\,\times \\ &\qquad\times\frac{n\!\!\!/_+}{2}(W_c^\dagger\xi)(z)_d\Big\}|0\rangle\,. \end{aligned} \tag{5.5}$$

We now insert (4.10) for the $\bar{B}$ meson matrix element of the soft fields. Only the colour component $T^A_{ba}T^A_{cd}$ contributes, and we are left with the term parameterized by the tri-local shape-function $\tilde{T}_2$. Going to momentum space shape-functions, we find

$$\begin{aligned} \langle\bar{B}|T^{\mu\nu}|\bar{B}\rangle &= \frac{2}{N_c^2-1}\frac{\epsilon^\perp_{\rho\sigma}}{2}\frac{1}{2}\left(\gamma^\nu\frac{1+v\!\!\!/}{2}\gamma^{\sigma_\perp}\gamma_5\gamma^\mu\right)_{\beta\alpha}H\left(\frac{n_+p}{m_b}\right)\times \\ &\quad\times\int d\omega_1 d\omega_2\, T_2(\omega_1,\omega_2)\int d^4xd^4z\, e^{ip\cdot x}\, e^{-i\omega_1x_+}\, e^{-i\omega_2z_+}\,(-i)z^{\rho_\perp}\,\times \\ &\quad\times in_-\partial_{(z)}\langle 0|\,T\Big\{\Big[(\bar{\xi}W_c)i\overleftarrow{\partial\!\!\!/}_\perp\Big](0)_\beta T^A(W_c^\dagger\xi)(x)_\alpha\Big(\bar{\xi}W_c\frac{n\!\!\!/_+}{2}T^AW_c^\dagger\xi\Big)(z)\Big\}|0\rangle\,. \end{aligned} \tag{5.6}$$

The integral over $x$ and $z$ defines the momentum space jet function $\mathcal{J}$. In the frame with $p_\perp = 0$ it can depend on the external momenta $n_+p$, $p^2$, and the convolution variables $\omega_{1,2}$, and the integral equals

$$\frac{N_c^2-1}{2}\left(\frac{\not{n}_-}{2}\gamma^{\rho_\perp}\right)_{\alpha\beta}\mathcal{J}(n_+p,p^2;\omega_1,\omega_2)\,. \tag{5.7}$$

Inserting this into (5.6) we obtain the final result

$$\begin{aligned}\langle \bar{B}|T^{\mu\nu}|\bar{B}\rangle &= \frac{\epsilon^\perp_{\rho\sigma}}{2}\,\frac{1}{2}\mathrm{tr}\left(\frac{\not{n}_-}{2}\gamma^{\rho_\perp}\gamma^\nu\frac{1+\not{v}}{2}\gamma^{\sigma_\perp}\gamma_5\gamma^\mu\right)\times\\ &\quad\times H\left(\frac{n_+p}{m_b}\right)\times\int d\omega_1 d\omega_2\, T_2(\omega_1,\omega_2)\mathcal{J}(n_+p,p^2;\omega_1,\omega_2)\,,\end{aligned} \tag{5.8}$$

which expresses the contribution from (5.1) to the current correlator as a product of a hard coefficient and a two-fold convolution of a shape-function and a jet-function.

It is instructive to examine the result in the tree approximation for the jet-function. In this approximation

$$\mathcal{J}^{\mathrm{tree}}(n_+p,p^2;\omega_1,\omega_2) = \omega_2\,\frac{in_+p}{p^2_{\omega_1}}\frac{in_+p}{p^2_{\omega_{12}}} = \frac{n_+p}{p^2_{\omega_1}} - \frac{n_+p}{p^2_{\omega_{12}}}\,, \tag{5.9}$$

where we used $n_+p\,\omega_2 = p^2_{\omega_1} - p^2_{\omega_{12}}$ ($p_{\omega_{1\ldots n}} = p - (\omega_1 + \cdots + \omega_n)n_+/2$). Each of the two terms after the second equality depends only on a single combination of the $\omega_{1,2}$, which allows us to write

$$\int d\omega_1 d\omega_2\, T_2(\omega_1,\omega_2)\,\mathcal{J}^{\mathrm{tree}}(n_+p,p^2;\omega_1,\omega_2) = \int d\omega\,\frac{n_+p}{p^2_\omega}\int d\omega'\Big(T_2(\omega-\omega',\omega') - T_2(\omega,\omega')\Big)\,. \tag{5.10}$$

The structure of this result implies in coordinate space that rather than the tri-local matrix element (4.10) parameterized by a function of two variables $\tilde{T}_2(x_+,z_+)$, the tree level approximation requires only the simpler combination $\tilde{T}_2(x_+,x_+) - \tilde{T}_2(x_+,0)$ related to the *bi-local* matrix element

$$\langle\bar{B}|(\bar{h}_v(-i)\overleftarrow{D}^s_{\mu_\perp}Y)(x_-)\gamma_{\nu_\perp}\gamma_5(Y^\dagger h_v)(0)|\bar{B}\rangle - \langle\bar{B}|(\bar{h}_vY)(x_-)\gamma_{\nu_\perp}\gamma_5(Y^\dagger iD^s_{\mu_\perp}h_v)(0)|\bar{B}\rangle\,. \tag{5.11}$$

The simplification of the tree level result is due to an equation of motion identity behind the cancellation of propagators in (5.9). We can see this directly in coordinate space by using the SCET equation of motion for the free-field propagator. Defining the contraction

$$\bar{\xi}(x)_{a\alpha}\xi(y)_{b\beta} = i\Delta(x-y)\,\delta_{ab}\left(\frac{\not{n}_-}{2}\right)_{\alpha\beta}\,, \tag{5.12}$$

the function $\Delta(z)$ satisfies

$$in_-\partial\Delta(z) = \delta^{(4)}(z) - \frac{(i\partial_\perp)^2}{in_+\partial}\Delta(z)\,. \tag{5.13}$$



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When this is used on the product of the two collinear propagators in (5.5), the second term on the right hand side of (5.13) gives zero, because we work in the frame $p_\perp = 0$, and the two delta-functions produce the two terms $\tilde{T}_2(x_+, x_+) - \tilde{T}_2(x_+, 0)$. We will see more of these simplifications when we work out the complete result in the tree approximation in the following section. In section 7 we ask in more generality whether the appearance of tetra- and tri-local shape-functions in the formalism could be spurious, and whether, perhaps, the final result could be expressed in terms of only bi- or tri-local matrix elements beyond the tree approximation.

## 6. Tree approximation

In this section we calculate the current correlator $T^{\mu\nu}$ (2.2) and the hadronic tensor in the tree approximation including the $1/m_b$ power corrections. In this approximation we can set the collinear gluon field $A_c$ to zero except in $\mathcal{L}^{(1)}_{\xi q}$, where we can draw a tree graph with external soft light quarks (see figure 1). The weak current $J_i = \bar{\psi}\Gamma_i Q$ including the hard coefficient functions (at tree level) is given up to order $\lambda^2$ by [8, 9]

$$J_i = J^{(0)} + \sum_{i=1}^{2} J_i^{(1)} + \sum_{i=1}^{4} J_i^{(2)} \tag{6.1}$$

with

$$J^{(0)} = \bar{\xi}\Gamma_i h_v\,,$$

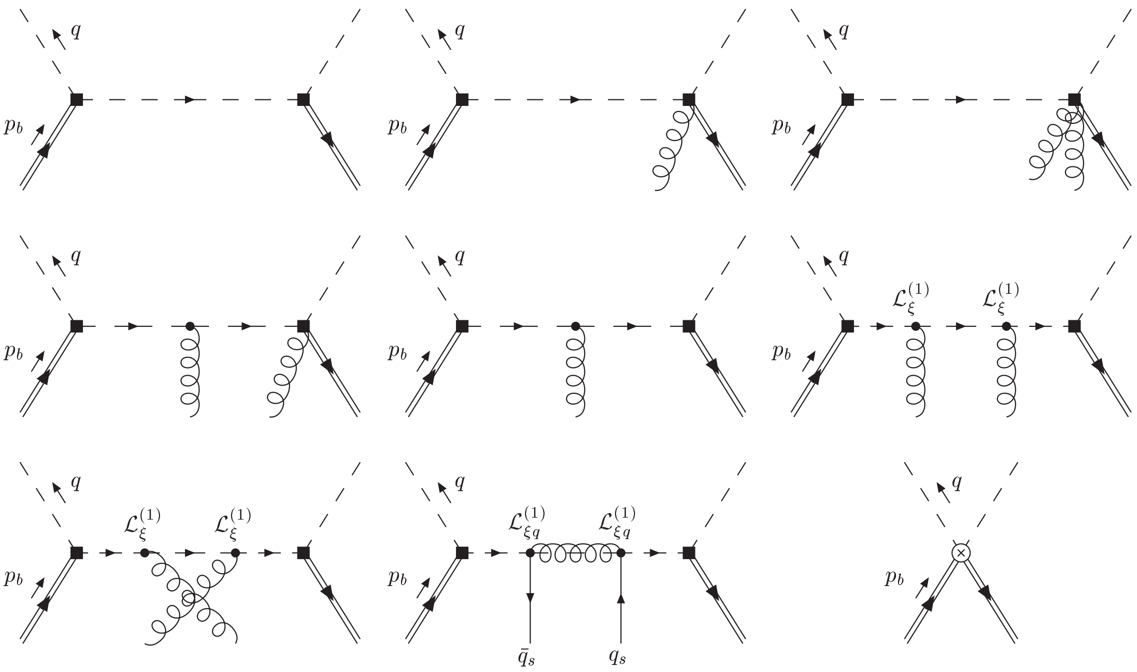


**Figure 1:** Tree diagrams contributing to the current correlator $T^{\mu\nu}$. Not shown are diagrams that vanish when $n_+A_c = 0$, $n_-A_s = 0$, or are symmetric to those shown.

$$J_1^{(1)} = \bar{\xi}\Gamma_i x_{\perp\mu} D^\mu_{\perp s} h_v\,, \qquad J_2^{(1)} = -\bar{\xi}\, i\overleftarrow{\not\partial}_\perp \frac{1}{in_+\overleftarrow{\partial}}\frac{\not n_+}{2}\Gamma_i h_v\,,$$
$$J_1^{(2)} = \bar{\xi}\Gamma_i \frac{n_- x}{2}\, n_+ D_s h_v\,, \qquad J_2^{(2)} = \bar{\xi}\Gamma_i \frac{x_{\mu\perp}x_{\nu\perp}}{2} D^\mu_{\perp s}\, D^\nu_{\perp s} h\,,$$
$$J_3^{(2)} = -\bar{\xi}\, i\overleftarrow{\not\partial}_\perp \frac{1}{in_+\overleftarrow{\partial}}\frac{\not n_+}{2}\Gamma_i x_{\perp\mu} D^\mu_{\perp s} h_v\,, \qquad J_4^{(2)} = \bar{\xi}\Gamma_i\, \frac{i\not D_s}{2m_b} h_v\,. \tag{6.2}$$

We now compute all the time-ordered products including insertions of sub-leading lagrangians that follow from the generic expressions (4.1) and the explicit form of the lagrangian (2.8), (2.9) and the currents (6.2). The diagrammatic representation of this computation is shown in figure 1.[2] The final result is rather compact, but the intermediate expressions among which many cancellations take place are lengthy. To display this result, we introduce a short-hand notation

$$J_a^\dagger\, J_b\, i\mathcal{L}_c \equiv i\int d^4x\, e^{ipx}\, T\left\{J_a^\dagger(x) J_b(0)\, i\int d^4z\, \mathcal{L}_c(z)\right\} \tag{6.3}$$

and similarly for the other terms. We also define the integral operators ($x_+ = (n_+x)/2$, $p_\perp = 0$, $n_+p > 0$)

$$\begin{aligned}
I_2^*\, f(x_+) &\equiv -\int d^4x\, e^{ipx}\int \frac{d^4k}{(2\pi)^4}\, e^{-ikx}\, \frac{n_+k}{k^2+i\epsilon}\, f(x_+) \\
&= -\int dx_+ e^{in_-px_+}\int \frac{dn_-k}{2\pi}\, e^{-in_-kx_+}\, \frac{1}{n_-k+i\epsilon}\, f(x_+) \\
&= i\int_0^\infty dx_+\, e^{in_-px_+}\, f(x_+)\,,
\end{aligned} \tag{6.4}$$

$$\begin{aligned}
I_3^*\, f(x_+,z_+) &\equiv -\int d^4x d^4z\, e^{ipx}\int \frac{d^4k}{(2\pi)^4}\frac{d^4k'}{(2\pi)^4}\, e^{-ik(x-z)}\,\times \\
&\quad \times e^{-ik'z}\, \frac{n_+k}{(k^2+i\epsilon)(k'^2+i\epsilon)}\, f(x_+,z_+) \\
&= I_2^*\, \frac{-i}{n_+p}\int_0^{x_+} dz_+\, f(x_+,z_+)\,,
\end{aligned} \tag{6.5}$$

and present the intermediate result in light-cone gauge $n_-A_s = 0$ ($Y_s = 1$). With this notation, the leading-power result for $T^{\mu\nu}$ reads in the tree approximation

$$J^{(0)\dagger}\, J^{(0)} = I_2^*\, \bar{h}_v(x_-)\Gamma_i^\dagger \frac{\not n_-}{2}\Gamma_j h_v(0). \tag{6.6}$$

Following the classification of time-ordered products in (4.1), we calculate the individual contributions and obtain

$$\begin{aligned}
(\mathrm{I}) &= J_1^{(2)\dagger} J^{(0)} = 0\,, \\
(\mathrm{II}) &= J_2^{(2)\dagger} J^{(0)} = -I_2^*\, \frac{1}{k^2}\, (\bar{h}_v(-i\overleftarrow{D}_{s\perp})^2)(x_-)\Gamma_i^\dagger \frac{\not n_-}{2}\Gamma_j h_v(0)\,,
\end{aligned}$$

[2]The last diagram in figure 1 is not reproduced by a correlation function of effective currents in SCET but requires an additional local term that we have not discussed. Since this term has no discontinuity it does not contribute to the hadronic tensor and decay distributions.



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$$
\begin{aligned}
(\mathrm{III}) &= J_3^{(2)\dagger} J^{(0)} = -I_2^* \,\frac{1}{n_+p}\,(\bar{h}_v(-i\overleftarrow{D}_s^{\mu_\perp}))(x_-)\Gamma_i^\dagger \frac{\not{n}_+}{2}\gamma_{\mu_\perp}\frac{\not{n}_-}{2}\Gamma_j h_v(0)\,, \\
(\mathrm{IV}) &= J_4^{(2)\dagger} J^{(0)} + J^{(0)\dagger} J_4^{(2)} \\
&= \frac{1}{2m_b}\, I_2^* \bigg( (\bar{h}_v(-i\overleftarrow{\not{D}}_s))(x_-)\Gamma_i^\dagger \frac{\not{n}_-}{2}\Gamma_j h_v(0) + \bar{h}_v(x_-)\Gamma_i^\dagger\frac{\not{n}_-}{2}\Gamma_j(i\not{D}_s h_v)(0)\bigg)\,, \\
(\mathrm{V}) &= J_1^{(1)\dagger} J_2^{(1)} = -I_2^* \,\frac{1}{n_+p}\,(\bar{h}_v(-i\overleftarrow{D}_s^{\mu_\perp}))(x_-)\Gamma_i^\dagger \frac{\not{n}_-}{2}\gamma_{\mu_\perp}\frac{\not{n}_+}{2}\Gamma_j h_v(0)\,, \\
(\mathrm{VI}) &= J_2^{(1)\dagger} J_2^{(1)} = 0\,, \\
(\mathrm{VII}) &= J_1^{(1)\dagger} J^{(0)}\, i\mathcal{L}_\xi^{(1)} = -I_3^*\, 2\, (\bar{h}_v(-i\overleftarrow{D}_s^{\mu_\perp}))(x_-)\Gamma_i^\dagger \frac{\not{n}_-}{2}\Gamma_j g A^s_{\mu_\perp}(z_-) h_v(0) \\
&\quad + I_2^*\, \frac{2}{k^2}\,(\bar{h}_v(-i\overleftarrow{D}_s^{\mu_\perp}) g A^s_{\mu_\perp})(x_-)\Gamma_i^\dagger\frac{\not{n}_-}{2}\Gamma_j h_v(0)\,, \\
(\mathrm{VIII}) &= (J_2^{(1)\dagger} J^{(0)} + J^{(0)\dagger} J_2^{(1)})\, i\mathcal{L}_\xi^{(1)} \\
&= I_2^*\,\frac{1}{n_+p}\bigg( (\bar{h}_v A_s^{\mu_\perp})(x_-)\Gamma_i^\dagger\frac{\not{n}_-}{2}\gamma_{\mu_\perp}\frac{\not{n}_+}{2}\Gamma_j h_v(0) - \bar{h}_v(x_-)\Gamma_i^\dagger\frac{\not{n}_-}{2}\gamma_{\mu_\perp}\frac{\not{n}_+}{2}\Gamma_j(A_s^{\mu_\perp}h_v)(0)\bigg)\,, \\
(\mathrm{IX}) &= J^{(0)\dagger} J^{(0)}\, i\mathcal{L}^{(2)}_{\mathrm{HQET}} = I_2^*\, T\left\{ \bar{h}_v(x_-)\Gamma_i^\dagger\frac{\not{n}_-}{2}\Gamma_j h_v(0)\, i\int d^4z\,\mathcal{L}^{(2)}_{\mathrm{HQET}}(z)\right\}\,, \\
(\mathrm{X}) &= J^{(0)\dagger} J^{(0)}\, i\mathcal{L}^{(2)}_{1\xi} = 0\,, \\
(\mathrm{XI}) &= J^{(0)\dagger} J^{(0)}\, i\mathcal{L}^{(2)}_{2\xi} = -I_3^*\,\frac{n_+p}{2}\, z^{\mu_\perp} z^{\lambda_\perp}\,\bar{h}_v(x_-)\Gamma_i^\dagger\frac{\not{n}_-}{2}\Gamma_j [D^s_{\lambda_\perp}, n_-^\nu g F^s_{\mu_\perp\nu}](z_-) h_v(0)\,, \\
(\mathrm{XII}) &= J^{(0)\dagger} J^{(0)}\, i\mathcal{L}^{(2)}_{3\xi} = I_3^*\,\frac{1}{2}\,\bar{h}_v(x_-)\Gamma_i^\dagger\frac{\not{n}_-}{2}\gamma_{\nu_\perp}\gamma_{\mu_\perp} ig F^s_{\mu_\perp\nu_\perp}(z_-)\Gamma_j h_v(0)\,, \\
(\mathrm{XIII}) &= \frac{1}{2}\, J^{(0)\dagger} J^{(0)}\, i\mathcal{L}_\xi^{(1)}\, i\mathcal{L}_\xi^{(1)} \\
&= -J^{(0)\dagger} J^{(0)}\, i\mathcal{L}^{(2)}_{2\xi} + I_3^*\, 2\,(\bar{h}_v g A_s^{\mu_\perp})(x_-)\Gamma_i^\dagger\frac{\not{n}_-}{2}\Gamma_j g A^s_{\mu_\perp}(z_-) h_v(0) - \\
&\quad - I_2^*\,\frac{2}{k^2}\,(\bar{h}_v g A_s^{\mu_\perp} g A^s_{\mu_\perp})(x_-)\Gamma_i^\dagger\frac{\not{n}_-}{2}\Gamma_j h_v(0) - \\
&\quad - I_3^*\,\bar{h}_v(x_-)\Gamma_i^\dagger\frac{\not{n}_-}{2}\Gamma_j(g A_s^{\mu_\perp} g A^s_{\mu_\perp} - [i\partial^{\mu_\perp} g A^s_{\mu_\perp}])(z_-) h_v(0) + \\
&\quad + I_2^*\,\frac{1}{k^2}\,(\bar{h}_v(g A_s^{\mu_\perp} g A^s_{\mu_\perp} - [i\partial^{\mu_\perp} g A^s_{\mu_\perp}]))(x_-)\Gamma_i^\dagger\frac{\not{n}_-}{2}\Gamma_j h_v(0)\,, \\
(\mathrm{XIV}) &= J^{(0)\dagger} J^{(0)}\, i\mathcal{L}^{(1)}_{\xi q}\, i\mathcal{L}^{(1)}_{\xi q} \\
&= \int d^4x d^4z_1 d^4z_2\, e^{ipx}\int \frac{d^4k}{(2\pi)^4}\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\, e^{-ik_1(x-z_2)}\, e^{-ik(z_2-z_1)}\, e^{-ik_2 z_1}\,\times \\
&\quad \times\frac{n_+k_1\, n_+k_2}{k^2 k_1^2 k_2^2}\, g^2\,\bar{h}_v(x_-)\Gamma_i^\dagger\frac{\not{n}_-}{2}\gamma^{\mu_\perp} T^A q_s(z_{2-})\,\bar{q}_s(z_{1-}) T^A\gamma_{\mu_\perp}\frac{\not{n}_-}{2}\Gamma_j h_v(0) \\
&= I_2^*\,\frac{1}{n_+p}\int_0^{x_+} dz_{1+}\int_{z_{1+}}^{x_+} dz_{2+}\, g^2\,\bar{h}_v(x_-)\Gamma_i^\dagger\,\times \\
&\quad \times\frac{\not{n}_-}{2}\gamma^{\mu_\perp} T^A q_s(z_{2-})\bar{q}_s(z_{1-}) T^A\gamma_{\mu_\perp}\frac{\not{n}_-}{2}\Gamma_j h_v(0)\,.
\end{aligned}
\tag{6.7}
$$

In order to arrive at this result we have made repeated use of the equation of motion (5.13) for the collinear propagator. The expressions simplify considerably after adding (III) + (V)

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+ (VIII) and (II) + (VII) + (XI) + (XII) + (XIII). Together with the other non-vanishing terms (IV), (IX) and (XIV) we obtain for the current correlation function at order $1/m_b$,

$$
\begin{aligned}
T^{1/m_b}_{\rm tree} = & -\int \frac{dx_+ d\omega}{2\pi}\, e^{i(n_-p-\omega)x_+} \frac{1}{\omega+i\epsilon} \times \\
& \times\Big\{ T\Big\{(\bar h_v Y)(x_-)\Gamma_i^\dagger \frac{\not n_-}{2}\Gamma_j (Y^\dagger h_v)(0)\; i\int d^4z\, \mathcal{L}^{(2)}_{\rm HQET}(z)\Big\} + \\
& + \frac{1}{2m_b}\Big[(\bar h_v(-i\overleftarrow{\not D}_s)Y)(x_-)\Gamma_i^\dagger \frac{\not n_-}{2}\Gamma_j(Y^\dagger h_v)(0) + \\
& \qquad + (\bar h_v Y)(x_-)\Gamma_i^\dagger\frac{\not n_-}{2}\Gamma_j(Y^\dagger i\not D_s h_v)(0)\Big] - \\
& - \frac{1}{n_+p}\Big[(\bar h_v(-i\overleftarrow{D}^{\mu_\perp}_s)Y)(x_-)\Gamma_i^\dagger \frac{\not n_+}{2}\gamma_{\mu_\perp}\frac{\not n_-}{2}\Gamma_j(Y^\dagger h_v)(0) + \\
& \qquad + (\bar h_v Y)(x_-)\Gamma_i^\dagger \frac{\not n_-}{2}\gamma_{\mu_\perp}\frac{\not n_+}{2}\Gamma_j(Y^\dagger(-i\overleftarrow{D}^{\mu_\perp}_s)h_v)(0)\Big] + \\
& + \frac{i}{n_+p}\int_0^{x_+} dz_+ (\bar h_v Y)(x_-)\Gamma_i^\dagger\frac{\not n_-}{2}(Y^\dagger(-i\overleftarrow{\not D}_{s\perp})(-i\overleftarrow{\not D}_{s\perp})Y)(z_-)\Gamma_j(Y^\dagger h_v)(0) \\
& + \frac{1}{n_+p}\int_0^{x_+}dz_{1+}\int_{z_{1+}}^{x_+} dz_{2+}\, g^2\, (\bar h_v Y)(x_-)\Gamma_i^\dagger \frac{\not n_-}{2}\gamma^{\mu_\perp}T^A(Y^\dagger q)(z_{2-})\times \\
& \times (\bar q Y)(z_{1-})T^A\gamma_{\mu_\perp}\frac{\not n_-}{2}\Gamma_j(Y^\dagger h_v)(0)\Big\}\,.
\end{aligned}
\tag{6.8}
$$

We have reinserted the soft Wilson lines $Y$, which makes this expression gauge invariant. The derivatives are understood to act on everything to their right (or to their left when indicated by the arrow) independent of the position argument of the field. It is worth noting that the double insertion of $\mathcal{L}^{(1)}_\xi$ (XIII) was expected to be tetra-local. However, only the delta-function terms survive in the application of the equation of motion identity and the complexity of this term is reduced to a tri-local term of the form $\bar h_v(x_-)[iD_\perp iD_\perp](z_-)h_v(0)$. On the other hand, the four-quark contribution from the double insertion of $\mathcal{L}^{(1)}_{\xi q}$ *is* tetra-local.

We have also performed the calculation in a general frame where $p_\perp \neq 0$, in which the result is a significantly more complicated expression. In particular, the double insertion of $\mathcal{L}^{(1)}_\xi$ gives a tetra-local term proportional to $p_\perp^2$. Superficially, this seems to require a much larger set of shape-functions, including a different degree of locality. However, the hadronic tensor depends only on two kinematic invariants $vp$ and $p^2$, so two out of the three variables $n_+p$, $n_-p$ and $p_\perp^2$ must be sufficient to reconstruct the complete information. We therefore conclude that the specific convolutions of jet-functions and tetra-local shape-functions from the double insertion of $\mathcal{L}^{(1)}_\xi$ (and other similar terms not present for $p_\perp = 0$) in the general frame cannot contain independent non-perturbative information despite their appearance. We can see this technically by noting that the transverse momentum is defined with respect to a choice of vectors $n_-$, $v$. A transverse Lorentz boost from a frame with $p_\perp = 0$ to a frame with $p_\perp \sim \lambda$ can be effected by a reparameterization of $n_- \to n_- + 2\epsilon_\perp - \epsilon_\perp^2 n_+$ with $\epsilon_\perp \sim -p_\perp/n_+p$ and $v$ fixed. The complete SCET expansion is invariant, because it reproduces a Lorentz-invariant theory, but the transformation reshuffles terms in the $\lambda$

expansion. In particular, the leading-order lagrangian changes $\mathcal{L}^{(0)} \to \mathcal{L}^{(0)} + \epsilon_\perp \delta\mathcal{L}^{(0)} + \cdots$. Accordingly for matrix elements

$$\langle O \rangle \to \langle O \rangle + \epsilon_\perp \int d^4z \, \langle T\{O \, i\delta\mathcal{L}^{(0)}(z)\}\rangle + \cdots , \tag{6.9}$$

from which it is clear that the leading-power bi-local term in the frame $p_\perp = 0$ gives rise to tetra-local terms proportional to $p_\perp^2$ at order $\lambda^2$ in the general frame. While this explains the structure of terms in the general frame, we did not verify explicitly the equivalence of the expressions, which seems to be a technical but unilluminating task.

We now proceed to the evaluation of the hadronic tensor (2.1). Starting from (6.8) this requires that we i) take the imaginary part, ii) take the $\bar{B}$ meson matrix element and iii) insert the decomposition of the soft matrix elements into scalar shape-functions as given in section 4. The imaginary part of (6.8) is obtained by the replacement

$$\frac{1}{\omega + i\epsilon} \to -\pi\delta(\omega) \, . \tag{6.10}$$

The $\bar{B}$ meson matrix elements can all be expressed in terms of the previously defined shape-functions, if the relation

$$\begin{aligned} &\langle \bar{B} | (\bar{h}_v Y)(x_-)_{a\alpha} (Y^\dagger i D_s^\mu Y)(z_-)_{cd} (Y^\dagger h_v)(0)_{b\beta} | \bar{B} \rangle = \\ &\quad = \langle \bar{B} | (\bar{h}_v Y)(-x_-)_{b\beta} (Y^\dagger (-i) \overleftarrow{D}_s^\mu Y)(z_- - x_-)_{dc} (Y^\dagger h_v)(0)_{a\alpha} | \bar{B} \rangle^* \end{aligned} \tag{6.11}$$

and $\partial_{\mu_\perp} \langle \bar{B} | \ldots | \bar{B} \rangle = 0$ is used. The scalar decomposition contains odd terms proportional to $\epsilon_{\mu\nu}^\perp$, which can be eliminated using $\epsilon_{\mu\nu}^\perp \gamma_\perp^\nu \gamma_5 = \gamma_{\mu_\perp} (\not{v} \not{n}_- - \not{n}_- \not{v})/2$. We then express the position space shape-functions in terms of their Fourier transforms defined as in (2.15) and perform the integrations over the positions left in (6.8). The result for the hadronic tensor including now the leading-power contribution (in the tree approximation) becomes

$$\begin{aligned} W^{\mu\nu} = \int d\omega \, \delta(n_- p - \omega) \Bigg\{ & \frac{1}{2} \mathrm{tr} \left[ P_+ \Gamma_i^\dagger \frac{\not{n}_-}{2} \Gamma_j \right] \times \\ & \times \Bigg( S(\omega) + \frac{1}{2m_b} \left[ s_{\mathrm{kin}}(\omega) + C_{\mathrm{mag}} \left( \frac{m_b}{\mu} \right) s_{\mathrm{mag}}(\omega) \right] - \\ & \quad - \frac{u_s(\omega) + 2v_s(\omega)}{n_+ p} \Bigg) + \\ & + \frac{1}{4m_b} \mathrm{tr} \left[ (\not{n}_- - \not{v}) \Gamma_i^\dagger \frac{\not{n}_-}{2} \Gamma_j \right] (\omega S(\omega) - t(\omega)) + \\ & + \frac{1}{4n_+ p} \mathrm{tr} \left[ P_+ \gamma_\perp^\alpha \gamma_5 \Gamma_i^\dagger \gamma_{\alpha\perp} \gamma_5 \Gamma_j \right] t(\omega) \\ & + \frac{1}{2n_+ p} \mathrm{tr} \left[ P_+ \not{n}_- \gamma_5 P_+ \Gamma_i^\dagger \frac{\not{n}_-}{2} \gamma_5 \Gamma_j \right] (u_a(\omega) - 2v_a(\omega)) \Bigg\} , \end{aligned} \tag{6.12}$$

where $P_+ = (1 + \not{v})/2$, and we introduced the definitions

$$u_s(\omega) = \int d\omega_1 d\omega_2 \, J_2(\omega; \omega_1, \omega_2) \, (u_1(\omega_1)\delta(\omega_2) + U_1(\omega_1, \omega_2)) \, ,$$

$$
\begin{aligned}
u_a(\omega) &= \int d\omega_1 d\omega_2\, J_2(\omega;\omega_1,\omega_2)\, U_3(\omega_1,\omega_2)\,, \\
v_s(\omega) &= \int d\omega_1 d\omega_2 d\omega_3\, J_3(\omega;\omega_1,\omega_3,\omega_2)\, g^2\, \mathcal{V}_{10}(\omega_1,\omega_2,\omega_3)\,, \\
v_a(\omega) &= \int d\omega_1 d\omega_2 d\omega_3\, J_3(\omega;\omega_1,\omega_3,\omega_2)\, g^2\, \mathcal{V}_{13}(\omega_1,\omega_2,\omega_3)\,, \qquad (6.13) \\
J_2(n_-p;\omega_1,\omega_2) &= -\frac{1}{\pi}\,\mathrm{Im}\,\frac{(n_+p)^2}{p^2_{\omega_1}p^2_{\omega_{12}}} = \frac{1}{\omega_2}(\delta(n_-p-\omega_1-\omega_2)-\delta(n_-p-\omega_1))\,, \\
J_3(n_-p;\omega_1,\omega_2,\omega_3) &= -\frac{1}{\pi}\,\mathrm{Im}\,\frac{(n_+p)^3}{p^2_{\omega_1}p^2_{\omega_{12}}p^2_{\omega_{123}}} \\
&= \frac{\delta(n_-p-\omega_1)}{\omega_2(\omega_2+\omega_3)} - \frac{\delta(n_-p-\omega_1-\omega_2)}{\omega_2\omega_3} + \frac{\delta(n_-p-\omega_1-\omega_2-\omega_3)}{\omega_3(\omega_2+\omega_3)} - \\
&\quad -\pi^2\delta(n_-p-\omega_1)\delta(\omega_2)\delta(\omega_3)\,, \qquad (6.14)
\end{aligned}
$$

and used that $t(\omega)$ is real [12]. The denominators in the definition of $J_2$ and $J_3$ are understood to be supplied with a principal value definition. Eq. (6.12) is our final result, valid for arbitrary Dirac structures $\Gamma_i$ and $\Gamma_j$ of the weak currents. Despite the fact that the hadronic matrix elements are tri- and tetra-local the final result can be written as single convolutions of integrated shape-functions. Hence, from a phenomenological point of view, the $1/m_b$ corrections can be parameterized by a set of functions depending only on a single variable, just as at leading power. We discuss in section 7 whether this holds when loop corrections are included. The factor $g^2$ in the definition of the integrated four-quark shape-functions $v_{s,a}(\omega)$ should not lead to the conclusion that these contributions are suppressed. In fact, the same factor of $g^2$ is also present in the tetra-local piece of the two-derivative matrix element (4.11), which is $g^2\bar{h}_v A_\perp A_\perp h_v$ in light-cone gauge, and here the $g^2$ is conventionally assumed to be normalized at the non-perturbative strong interaction scale.

The $\bar{B}$ decay distributions are most conveniently expressed in terms of the scalar components of the hadronic tensor. In the following we give the result for the currents relevant to semi-leptonic $\bar{B} \to X_u \ell\bar{\nu}$ decay and the radiative $\bar{B} \to X_s\gamma$ decay in the convention specified by (2.3). For the semi-leptonic decay

$$
\Gamma_i^\dagger = \gamma^\mu(1-\gamma_5)\,, \qquad \Gamma_j = \gamma^\nu(1-\gamma_5)\,, \qquad (6.15)
$$

and we find

$$
\begin{aligned}
W_1 &= \frac{2}{n_+p}\int d\omega\,\delta(n_-p-\omega)\left[\left(1+\frac{n_-p}{n_+p}\right)S(\omega) + \frac{s_{\rm kin}(\omega)+C_{\rm mag}(m_b/\mu)s_{\rm mag}(\omega)}{2m_b} - \right. \\
&\qquad \left. - \frac{\omega S(\omega)-t(\omega)}{m_b} - \frac{u_s(\omega)+u_a(\omega)}{n_+p} - \frac{2(v_s(\omega)-v_a(\omega))}{n_+p}\right], \\
W_2 &= \frac{1}{2}W_3 = -\frac{2n_-p}{n_+p}\int d\omega\,\delta(n_-p-\omega)\,S(\omega)\,, \\
W_4 &= -\frac{4}{(n_+p)^2}\int d\omega\,\delta(n_-p-\omega)\,t(\omega)\,, \\
W_5 &= \frac{8}{(n_+p)^2}\int d\omega\,\delta(n_-p-\omega)\left[\frac{\omega S(\omega)-t(\omega)}{m_b} + \frac{t(\omega)+u_a(\omega)-2v_a(\omega)}{n_+p}\right]. \qquad (6.16)
\end{aligned}
$$

The result is given in the frame $p_\perp = 0$, where $vp = (n_+p + n_-p)/2$ and $p^2 = n_+p\, n_-p$. This can be used to convert the expressions to hadronic variables, see (2.4). We recall that the expressions are valid in the kinematic region where $n_+p \sim m_b$ and $n_-p \sim \Lambda_{\rm QCD}$. Note that only $W_1$ has a leading-power term.

Turning to the radiative decay $\bar{B} \to X_s\gamma$, we note that the photon momentum is $q = E_\gamma n_+$ in the frame $p_\perp = -q_\perp = 0$. Neglecting terms proportional to $q^\mu$ which vanish when the hadronic tensor is contracted with the photon polarization vector, we can take the current to be

$$\Gamma_i^\dagger = \frac{1}{4}\,[\not{n}_+, \gamma_\perp^\mu](1-\gamma_5)\,, \qquad \Gamma_j = \frac{1}{4}\,[\gamma_\perp^\nu, \not{n}_+](1+\gamma_5)\,. \tag{6.17}$$

It is simpler here not to decompose the hadronic tensor into scalar functions. Evaluating the traces in (6.12) we obtain

$$\begin{aligned} W^{\mu\nu} \;=\; & -(g_\perp^{\mu\nu} + \epsilon_\perp^{\mu\nu}) \int d\omega\, \delta(n_-p-\omega) \Bigg\{ S(\omega) + \frac{s_{\rm kin}(\omega) + C_{\rm mag}(m_b/\mu) s_{\rm mag}(\omega)}{2m_b} + \\ & + \frac{\omega S(\omega) - t(\omega)}{m_b} - \frac{u_s(\omega)+u_a(\omega)}{n_+p} - \frac{2(v_s(\omega)-v_a(\omega))}{n_+p} \Bigg\}. \end{aligned} \tag{6.18}$$

When the polarization of the photon is not observed, the $\epsilon_\perp^{\mu\nu}$ term does not contribute and the photon energy spectrum reads

$$\begin{aligned} \frac{1}{2\Gamma}\frac{d\Gamma}{dE_\gamma} = & \left(1 - \frac{2n_-p}{m_b}\right) S(n_-p) + \frac{s_{\rm kin}(n_-p) + C_{\rm mag}(m_b/\mu) s_{\rm mag}(n_-p)}{2m_b} - \\ & - \frac{1}{m_b}\Big( t(n_-p) + u_s(n_-p) + u_a(n_-p) + 2(v_s(n_-p) - v_a(n_-p)\Big) \end{aligned} \tag{6.19}$$

with $n_-p = m_b - 2E_\gamma$. This result is valid at tree level, and in the approximation where the four-quark operators in the weak effective hamiltonian are neglected.[3]

The hadronic tensor involves a power correction $(s_{\rm kin}(\omega) + C_{\rm mag}(m_b/\mu) s_{\rm mag}(\omega))/(2m_b)$ from the insertion of the $1/m_b$ corrections to the HQET lagrangian, $\mathcal{L}^{(2)}_{\rm HQET}$, because it is conventional to evaluate the soft matrix element with the leading-power HQET lagrangian. This is advantageous in applications of HQET, where use is made of the heavy-quark spin-flavour symmetries. In the present case it is more convenient to not treat the power corrections to the HQET lagrangian in the interaction picture. Then the above-mentioned term should be omitted, but the matrix elements $\langle \bar{B}|\ldots|\bar{B}\rangle$ are evaluated with the exact HQET lagrangian including $\mathcal{L}^{(2)}_{\rm HQET}$. In this picture the HQET matrix elements have a (small) $m_b$ dependence, but this is not an issue as long as all corrections up to a required order are included. It is further useful to regard the shape-functions as functions of the *hadronic* variable $n_-P = n_-p + M_B - m_b$ ($P$ is the total momentum of the hadronic final state), i.e. for any function $f(\omega)$ above we define $f(\omega) = \hat{f}(\omega + M_B - m_b)$, such that a

[3]There is no tree level contribution from four-quark operators at leading power. At order $1/m_b$ a non-zero contribution arises from soft gluons attached to the charm quark loop. The degree of non-locality of these terms depends on whether an expansion in $1/m_c$ is performed.

physical spectrum such as (6.19) is expressed in terms of $\hat{f}(n_-P)$. Note that $M_B - m_b$ is *not* the HQET parameter $\bar{\Lambda}$, but the difference between the physical meson mass and the heavy quark mass. The distinction is relevant at the level of power corrections to the spectrum. The advantage of taking the matrix elements with respect to the exact HQET lagrangian is that the support of the functions $\hat{f}$ is then from 0 to $\infty$.[4]

Comparing our result with previous work [12, 13, 15], we find agreement for the photon energy spectrum in the radiative decay [13] and the hadronic invariant mass spectrum in the semi-leptonic decay [15], provided we neglect the effect from the four-quark shape-functions $v_{s,a}(\omega)$. However, our general result (6.8) and the lepton energy spectrum in the semi-leptonic decay following from this result differ from [13] even when the four-quark shape-functions are neglected. The short-distance expansion of the hadronic tensor was obtained in previous work by direct matching of QCD to heavy quark effective theory without the intermediate use of soft-collinear effective theory. In the tree approximation the two approaches should give the same result. However, some of the results in [12, 13] are obtained by an expansion in transverse momentum of a hadronic tensor that effectively includes the integration over neutrino momentum. This can lead to incorrect results, because the transverse momentum relevant to the expansion is the transverse momentum of partons relative to the jet, not the jet and the neutrino. The calculations of the photon energy spectrum in the radiative decay and of the hadronic invariant mass spectrum in the semi-leptonic decay are not affected by this problem.

## 7. Remarks on factorization beyond tree level

We have just seen that at tree level many simplifications take place that reduce the degree of non-locality of shape-functions appearing in the $1/m_b$ corrections. In this section we investigate whether this simplification persists beyond tree level. Since the conclusion will be negative, it suffices to illustrate this point for the case of abelian gauge fields.

The manipulations below rely on the analogue of the QED Ward identity for the leading power (abelian) SCET lagrangian after the collinear field redefinition that decouples soft and collinear fields. Defining

$$
\begin{aligned}
J_+ &= \bar{\xi}\frac{\not{n}_+}{2}\xi\,, \\
J_\perp^\mu &= \bar{\xi}\left(i\overleftarrow{\not{D}}_{\perp c}\frac{1}{in_+\overleftarrow{D}_c}\gamma^{\mu_\perp} + \gamma^{\mu_\perp}\frac{1}{in_+D_c}i\not{D}_{\perp c}\right)\frac{\not{n}_+}{2}\xi\,, \\
J_- &= \bar{\xi}i\overleftarrow{\not{D}}_{\perp c}\frac{1}{in_+\overleftarrow{D}_c}\frac{\not{n}_+}{2}\frac{1}{in_+D_c}i\not{D}_{\perp c}\xi\,,
\end{aligned}
\tag{7.1}
$$

the Ward identity for jet-functions reads

$$
\begin{aligned}
&n_-\partial_{(z)}\langle 0|T\Big\{J_+(z)\chi(x_1)\ldots\chi(x_m)\Big\}|0\rangle + \\
&+\partial_{(z)}^{\mu_\perp}\langle 0|T\Big\{J_{\perp\mu}(z)\chi(x_1)\ldots\chi(x_m)\Big\}|0\rangle +
\end{aligned}
$$

[4] The upper limit is infinity as a consequence of factorization, which removes the physical upper limit and replaces it by a cut-off. However, dimensional regularization does not provide a dimensionful cut-off.

$$+n_+\partial_{(z)}\langle 0|T\Big\{J_-(z)\chi(x_1)\ldots\chi(x_m)\Big\}|0\rangle = -\sum_{k=1}^{m}(-1)^P\,\delta^{(4)}(z-x_k)\times$$
$$\times\langle 0|T\Big\{\chi(x_1)\ldots\chi(x_m)\Big\}|0\rangle\,, \qquad (7.2)$$

where $P=0$ if $\chi=\xi$ and $P=1$ if $\chi=\bar\xi$. This generalizes to insertions of composite operators by taking the local limit of a product of fundamental fields. (For derivative operators some of the delta-functions terms will then be derivatives of delta-functions.) Our main application of the Ward identity will be in the form

$$\begin{aligned}
&\int d^4z\, f(z_+,z_\perp)\,n_-\partial_{(z)}\langle 0|T\Big\{(\bar\xi\frac{\not{n}_+}{2}\xi)(z)\chi(x_1)\ldots\chi(x_m)\Big\}|0\rangle = \\
&= \int d^4z\,[\partial_{\mu_\perp}f(z_+,z_\perp)]\,\langle 0|T\Big\{J_\perp^\mu(z)\chi(x_1)\ldots\chi(x_m)\Big\}|0\rangle - \\
&- \sum_{k=1}^{m}(-1)^P\,f(x_{k+},x_{k\perp})\langle 0|T\Big\{\chi(x_1)\ldots\chi(x_m)\Big\}|0\rangle\,. \qquad (7.3)
\end{aligned}$$

### 7.1 Degree of multi-locality

The highest degree of non-locality in the product of soft fields at order $1/m_b$ comes from the double insertions of $\mathcal{L}_\xi^{(1)}$ and $\mathcal{L}_{\xi q}^{(1)}$. The double insertion of the mixed collinear-soft quark lagrangian $\mathcal{L}_{\xi q}^{(1)}$ is tetra-local even at tree level. On the other hand

$$\langle\bar B|T\Big\{(\bar h_v\Gamma_i^\dagger W_c^\dagger\xi)(x)(\bar\xi W_c\Gamma_j h_v)(0)\frac{1}{2}i\int d^4z_1\,\mathcal{L}_\xi^{(1)}(z_1)\,i\int d^4z_2\,\mathcal{L}_\xi^{(1)}(z_2)\Big\}|\bar B\rangle\,, \qquad (7.4)$$

while superficially tetra-local, actually reduces to a tri-local term at tree level. Does this hold to all orders?

In the abelian theory the collinear Wilson lines in $\mathcal{L}_\xi^{(1)}$ drop out. We use (4.6) and integrate by parts to write (after the collinear-soft decoupling field redefinition)

$$i\int d^4z\,\mathcal{L}_\xi^{(1)}(z) = \int d^4z\,z^{\mu_\perp}\,(Y^\dagger[iD^s_{\mu_\perp}Y])(z_-)\,in_-\partial_{(z)}\Big(\bar\xi\frac{\not{n}_+}{2}\xi\Big)(z)\,. \qquad (7.5)$$

Applying the Ward identity twice to (7.4), we can cast this contribution into the form

$$\begin{aligned}
&-\frac{1}{2}\int d^4z_1\int d^4z_2\,\langle\bar B|(\bar h_vY\Gamma_i^\dagger)(x_-)_\alpha(Y^\dagger[iD^s_{\mu_\perp}Y])(z_{1-})\times \\
&\qquad (Y^\dagger[iD^s_{\nu_\perp}Y])(z_{2-})(\Gamma_jY^\dagger h_v)(0)_\beta|\bar B\rangle\times \\
&\times\langle 0|T\Big\{(W_c^\dagger\xi)(x)_\alpha(\bar\xi W_c)(0)_\beta\,J_\perp^\mu(z_1)\,J_\perp^\nu(z_2)\Big\}|0\rangle + \delta\text{-function terms}\,, \qquad (7.6)
\end{aligned}$$

where "$\delta$-function terms" refers to contributions from the Ward identity which are only bi- or tri-local. In this form it is evident that the tetra-local term vanishes at tree level, because the jet-function with two insertions of the transverse current integrated over the transverse positions vanishes in this approximation. In momentum space, the derivative in the definition of $J_\perp^\mu$ must be proportional to the transverse component of a collinear vector, but in the frame where $p_\perp=0$ there is no such vector. By the same line of reasoning the jet-function cannot be expected to be zero beyond tree level. When the insertion of the current appears inside a collinear loop diagram, the transverse derivative produces factors of transverse loop momentum and an even number of such factors results in a $g_\perp^{\mu\nu}$ term after integration. We therefore conclude that beyond tree level tetra-local terms appear also from the double insertions of the sub-leading collinear lagrangian, and the complexity of the expression is not reduced.

### 7.2 Transformation to the covariant derivative basis

The SCET lagrangian has many terms in which the combination $x_\perp^\mu n_-^\nu g F_{\mu\nu}(x_-)$ occurs. These can be removed by applying the identity (4.6) (see also (7.5)). At tree level we have seen that the various terms organize themselves somewhat miraculously into covariant derivatives $(Y^\dagger i D_s^\mu Y)(z)$ despite the fact that this object represents two separate entities (see the two terms on the right hand side of (4.7)). For this reason we have chosen the covariant derivative basis for the soft matrix elements in section 4.

Using the Ward identity it is straightforward to show that at order $\lambda$, the two terms $J_1^{(1)\dagger} J^{(0)}$ and $J^{(0)\dagger} J^{(0)}\, i\mathcal{L}_\xi^{(1)}$ can be combined into an expression containing a soft matrix element with a single covariant derivative (as in (4.10)). At order $\lambda^2$, we find (using the short-hand notation of section 6)

$$
\begin{aligned}
&(\mathrm{II}) + (\mathrm{VII}) + (\mathrm{XI}) + (\mathrm{XII}) = \\
&\quad = J_2^{(2)\dagger} J^{(0)} + J_1^{(1)\dagger} J^{(0)}\, i\mathcal{L}_\xi^{(1)} + J^{(0)\dagger} J^{(0)}\, i\mathcal{L}_{2\xi}^{(2)} + J^{(0)\dagger} J^{(0)}\, i\mathcal{L}_\xi^{(1)}\, i\mathcal{L}_\xi^{(1)} \\
&\quad = \frac{1}{4}\mathrm{tr}\left[P_+ \Gamma_i^\dagger \frac{\not{n}_-}{2} \Gamma_j\right] i \int d^4x\, e^{ipx} \times \\
&\quad \times \bigg\{ \int d^4z_1 \int d^4z_2\, \langle \bar{B}|(\bar{h}_v Y)(x_-)(Y^\dagger i D^s_{\mu\perp} Y)(z_{1-})(Y^\dagger i D^s_{\nu\perp} Y)(z_{2-})(Y^\dagger h_v)(0)|\bar{B}\rangle \times \\
&\qquad \times \langle 0|T\Big\{(\bar{\xi} W_c)(0)(W_c^\dagger \xi)(x)\, J_\perp^\mu(z_1)\, J_\perp^\nu(z_2)\Big\}|0\rangle - \\
&\qquad - 2i \int d^4z\, \langle \bar{B}|(\bar{h}_v Y)(x_-)(Y^\dagger (i D^s_\perp)^2\, Y)(z_-)(Y^\dagger h_v)(0)|\bar{B}\rangle \times \\
&\qquad \times \langle 0|T\Big\{(\bar{\xi} W_c)(0)(W_c^\dagger \xi)(x)\, (\bar{\xi} \frac{1}{i n_+ D_c} \frac{\not{n}_+}{2} \xi)(z)\Big\}|0\rangle + \cdots \bigg\},
\end{aligned} \tag{7.7}
$$

where the ellipses denote further terms involving $\partial_\perp A_\perp$. Here we recognize the tetra-local shape-function with two covariant derivatives defined in (4.11). As mentioned above and seen explicitly in section 6 this term is multiplied by a jet-function that is zero at tree level in the frame $p_\perp = 0$. The second term does not vanish at tree level and reproduces the corresponding terms in section 6.

It therefore appears that in general the structure of terms that follows from the currents and lagrangian insertions of SCET can be systematically simplified by eliminating all the terms that involve factors of position from the multipole expansion. Application of the Ward identities then organizes the terms into covariant derivatives as displayed above. While we did not prove this statement in general, and in particular for the non-abelian case, we may note that when the hybrid momentum-position space formulation of SCET is employed [6], we are led directly to an expansion in terms of covariant derivatives.



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### 7.3 Convolutions with "effective" shape-functions

In the tree approximation the hadronic tensor (6.12) can be written in terms of the "effective" shape-functions defined in (6.13) that depend only on a single variable. This seems to be a simplification, since the information about hadronic matrix elements is encoded in a set of single- rather than multi-variable functions.

This can always be done for contributions to the hadronic tensor from two-body currents in SCET. In these cases $n_+p$ is the only large collinear momentum on which the jet-function can depend, and Lorentz invariance and dimensional analysis imply that the dependence on $n_+p$ can be factored out except for powers of logarithms of $n_+p/\mu$. The dependence on the remaining external momentum $n_-p$ can always be formally decoupled by introducing $\int d\omega\, \delta(n_-p-\omega)$ to define effective shape-functions that depend on only one variable $\omega$. The so-defined effective shape-functions are in fact convolutions of jet-functions and soft matrix elements as exhibited in (6.13) in the tree approximation. However, beyond this approximation there are contributions from three- and four-body currents as given in section 3. The jet-functions are then convolutions of a number of large collinear momentum components of the effective vertices and the dependence on $n_+p$ can no longer be factored, but arises after the convolution with hard coefficient functions. Even restricting attention to the contributions from two-body currents, the definitions of effective shape-functions such $u_{s,a}(\omega)$ and $v_{s,a}(\omega)$ in (6.13) are useful only in the tree approximation, because the weight functions (jet-functions) $J_{2,3}$ change at order $\alpha_s$. In any case, since the number of functions exceeds the number of observables, one must return to their definitions in terms of multi-local hadronic matrix elements in order to obtain an estimate from a particular hadronic model.

## 8. Estimate of four-quark contributions

In this section we shall make a few remarks concerning the phenomenological implications of our results. We do not perform an exhaustive analysis, rather we restrict ourselves to the new contribution from the four-quark shape-functions, which has not been studied in [12, 13]. The contribution we are going to consider involves $\bar{B}$ meson matrix elements $\langle \bar{B}|\bar{h}_v(x_-)h_v(0)\bar{q}(z_{1-})q(z_{2-})|\bar{B}\rangle$ of two heavy quark and two soft light quark fields, all at different positions on the light-cone. An interesting aspect of this contribution is that the soft quark flavour originates from the weak decay vertex. For semi-leptonic $\bar{B} \to X_u\ell\bar{\nu}$ decay it has to be a $u$ quark. The matrix element is therefore expected to be different for the decay of a charged $\bar{B}$ meson (where the flavour of the spectator quark matches $u$) and a neutral $\bar{B}$ meson. As we have seen, this effect is of order $1/m_b$ in the decay spectra, while such differences in the total semi-leptonic rates between charged and neutral $\bar{B}$ mesons are at least of order $1/m_b^3$ [14, 21].

The general expression for the four-quark shape-function has been given in (4.12). It involves 16 scalar functions, but only two of these, $\mathcal{V}_{10}$ and $\mathcal{V}_{13}$, appear in the hadronic tensors of the inclusive decays $\bar{B} \to X_s\gamma$ and $\bar{B} \to X_u\ell\bar{\nu}$ at tree level. These functions are tetra-local and their Fourier transforms depend on three variables. However, the result can be written in terms of two effective shape-functions $v_{s,a}(\omega)$, see (6.13).

In order to obtain a quantitative estimate of these contributions we have to employ a model, which at present can be only quite crude. Matrix elements of four-quark operators are often approximated in the "vacuum saturation approximation," where (leaving out spin and colour indices)

$$\langle\bar{B}|\bar{h}_v(x_-)u(z_{2-})\bar{u}(z_{1-})h_v(0)|\bar{B}\rangle \quad \rightarrow \quad \langle\bar{B}|\bar{h}_v(x_-)u(z_{2-})|0\rangle\langle 0|\bar{u}(z_{1-})h_v(0)|\bar{B}\rangle\,. \tag{8.1}$$

This gives $\mathcal{V}_{9-16} = 0$, since for these functions the heavy and light quark are in a colour-octet configuration. We conclude that the tree level contributions are suppressed, and that we need a better model for a quantitative estimate.

To account for the suppression of the colour-octet matrix elements, we introduce a parameter $\epsilon$ and write (now including colour)

$$\begin{aligned}&\langle\bar{B}|\bar{h}_v(x_-)T^A u(z_{2-})\bar{u}(z_{1-})T^A h_v(0)|\bar{B}\rangle\\ &\qquad \rightarrow \quad \varepsilon\,\langle\bar{B}|\bar{h}_v(x_-)u(z_{2-})|0\rangle\langle 0|\bar{u}(z_{1-})h_v(0)|\bar{B}\rangle.\end{aligned} \tag{8.2}$$

For local four-quark operators the agreement of calculations of the lifetime difference of the $B^-$ and $\bar{B}^0$ mesons [22, 23] with observations suggests that $|\varepsilon|$ cannot be larger than 0.1 without invoking large cancellations between different terms. Assuming (8.2) we can express the four-quark shape-functions in terms of the square of the $\bar{B}$ meson light-cone distribution amplitude. The resulting parameterizations of $v_{s,a}(\omega)$ exhibit a delta-function singularity at $\omega = m_b - M_B$, which can be traced to the fact that in the vacuum saturation approximation no dynamical (soft) momentum is transferred between the $\bar{h}_v u$ and the $\bar{u}h_v$ configuration. This is clearly not the case in reality, where soft gluons must be exchanged simply to rearrange colour. This motivates our "modified vacuum saturation approximation," where we allow the intermediate state to carry soft momentum $k$, and write

$$\begin{aligned}&\langle\bar{B}|\bar{h}_v(x_-)T^A u(z_{2-})\bar{u}(z_{1-})T^A h_v(0)|\bar{B}\rangle\\ &\qquad \rightarrow \quad \varepsilon \int dk_+ f(k_+)\langle\bar{B}|\bar{h}_v(x_-)u(z_{2-})|k\rangle\langle k|\bar{u}(z_{1-})h_v(0)|\bar{B}\rangle\,,\end{aligned} \tag{8.3}$$

where $|k\rangle$ represents a soft state of momentum $k$ and $k_+ = n_- k$. $f(k_+)$ (whose integral is normalized to 1) represents a (non-perturbative) distribution function of the $k_+$ momentum component of the soft state. We then write

$$\begin{aligned}\langle k|\bar{u}(y_-)_\alpha h_v(x_-)_\beta|\bar{B}\rangle &= e^{-i(M_B-m_b-k_+)x_+}\,\langle k|\bar{u}(y_- - x_-)_\alpha h_v(0)_\beta|\bar{B}\rangle\\ &\approx e^{-i(M_B-m_b-k_+)x_+}\,\langle 0|\bar{u}(y_- - x_-)_\alpha h_v(0)_\beta|\bar{B}\rangle\\ &= e^{-i(M_B-m_b-k_+)x_+}\,\frac{(-i)f_B M_B}{2\sqrt{2M_B}}\left(\frac{1+v}{2}\gamma_5\right)_{\beta\alpha}\tilde{\phi}_{B+}(t) + \not{n}_- - \text{term}\end{aligned} \tag{8.4}$$

with $t = (y-x)_+$. The transition to the second line involves the assumption that the $t$-dependence of the matrix element is not modified by the "soft" state. This may be difficult to justify, but it allows us to obtain the desired matrix element in terms of the light-cone distribution amplitude of the $\bar{B}$ meson [24, 25] about which a few things are known. A second possible structure proportional to $\not{n}_-$ is not given above, because it drops out in the following calculations. The momentum space distribution amplitude is defined by

$$\tilde{\phi}_{B_+}(t) = \int_0^\infty d\omega\, e^{-i\omega t}\,\phi_{B+}(\omega)\,. \tag{8.5}$$

It is now straightforward to insert (8.4) into (8.3) and to match the Dirac structure to those defining the general decomposition of the four-quark shape-function. We then find that the two relevant effective shape-functions are given by

$$\begin{aligned} v_s(n_-p) &= -v_a(n_-p) \\ &= \frac{\pi\alpha_s}{4}\, f_B^2 M_B\,\varepsilon \int dk_+\, f(k_+)\,\times \\ &\quad\times\left\{\delta(n_-P-k_+)\left[\int_0^\infty \frac{d\omega}{\omega}\,\phi_{B+}(\omega)\right]^2 + \right. \\ &\qquad\left. +\,\frac{2\phi_{B+}(n_-P-k_+)}{n_-P-k_+}\,P\int_0^\infty d\omega\,\frac{\phi_{B+}(\omega)}{n_-P-\omega-k_+}\right\} \\ &= \frac{\pi\alpha_s}{4}\,\frac{f_B^2 M_B}{\lambda_B^2}\,\varepsilon f(n_-P)\,\times \\ &\quad\times\left\{1+\frac{2\lambda_B^2}{f(n_-P)}\int_0^{n_-P} dk_+\, f(n_-P-k_+)\,\frac{\phi_{B+}(k_+)}{k_+} P\int_0^\infty d\omega\,\frac{\phi_{B+}(\omega)}{k_+-\omega}\right\}, \end{aligned} \tag{8.6}$$

where $P$ denotes the principal value and $1/\lambda_B = \int_0^\infty d\omega\,\phi_{B+}(\omega)/\omega$. We observe that the final result is expressed in terms of the hadronic variable $n_-P \equiv P_+ = E_H - |\vec{P}_H|$. We can get an idea of the magnitude of the power correction from the four-quark shape-functions from (6.12), which instructs us to compare $4v_s(n_-p)/n_+p$ to $S(n_-p) \equiv \hat{S}(n_-p+M_B-m_b) = \hat{S}(n_-P)$. The integration over $n_+p$ replaces $n_+p$ by an average value $\langle n_+p\rangle$ of order $M_B$, hence

$$\frac{4v_s(n_-p)}{\langle n_+p\rangle \hat{S}(n_-P)} \approx \pi\alpha_s \frac{f_B^2}{\lambda_B^2}\,\varepsilon\,\frac{M_B}{\langle n_+p\rangle}\,\frac{f(n_-P)}{\hat{S}(n_-P)}\{1+\cdots\}\,, \tag{8.7}$$

where the ellipses in brackets stand for the twofold integral in (8.6). If we set the bracket to one and assume that $f$ and $\hat{S}$ have similar shapes, we see that the size of the correction is about

$$\pi\alpha_s \frac{f_B^2}{\lambda_B^2}\,\varepsilon \approx \pm 5\%\,, \tag{8.8}$$

compatible with a power correction. Evidently there is a large uncertainty associated with this estimate, since, besides the assumptions in the construction of the model, $\alpha_s$ could be anything between 0.3 (corresponding to the hard-collinear scale) and 1 (corresponding to the soft scale) and $|\varepsilon| = 0.1$ is just a guess. We should emphasize that the estimate is for the semi-leptonic decay of a $B^-$. For $\bar{B}^0$ the correction vanishes in our approximation. The distinctive feature of the four-quark contributions is therefore a difference in the $B^-$ and $\bar{B}^0$ spectra, which could be of order 5%.

The effect could be more important than the estimate suggests, if it leads to a significant distortion of the spectrum in the region of $n_-P \sim \Lambda_{\rm QCD}$. For instance, if $f(n_-P)$

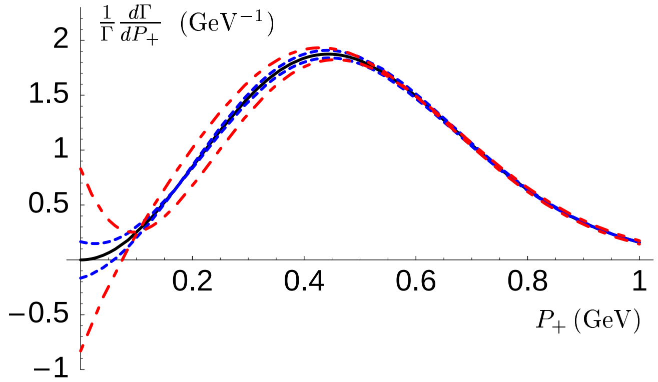


**Figure 2:** Distortion of the $P_+$ spectrum in $B^- \to X_u \ell\nu$ decay by four-quark contributions assuming the model (8.9). The solid central curve is $\hat{S}(P_+)$. The dashed curves correspond to $\lambda = 100\,\mathrm{MeV}$ (long dashes) and $\lambda = 500\,\mathrm{MeV}$ (short dashes). Each pair is for $\varepsilon = \pm 0.1$.

were peaked at smaller arguments than $\hat{S}(n_-P)$, as would be the case in the naive vacuum saturation approximation, where the "soft" intermediate state has no momentum, the four-quark contribution could give a significant enhancement at small values of $n_-P$. To be definite, let us assume the following models for the various functions:

$$\begin{aligned}\hat{S}(\omega) &= \frac{32\omega^2}{\pi^2\bar{\Lambda}^3}\exp\left[-\frac{4\omega^2}{\pi\bar{\Lambda}^2}\right],\\ \phi_{B+}(\omega) &= \left(\frac{3}{2\bar{\Lambda}}\right)^2 \omega e^{-3\omega/2\bar{\Lambda}},\\ f(k_+) &= \frac{1}{\lambda}\,e^{-k_+/\lambda}.\end{aligned} \tag{8.9}$$

To display the effect on a decay spectrum, it is natural to consider the distribution in the hadronic light-cone variable $P_+ = E_H - |\vec{P}_H|$, since it is proportional to $\hat{S}(P_+)$ at leading power. Including the four-quark shape-functions but neglecting the other power corrections, we obtain for $B^- \to X_u\ell\nu$

$$\frac{1}{\Gamma}\frac{d\Gamma}{dP_+} = \hat{S}(P_+) - \frac{10v_s(n_-p) + 2v_a(n_-p)}{3m_b}\,. \tag{8.10}$$

In figure 2 we show the effect of the four-quark contribution on the spectrum of $B^- \to X_u\ell\nu$ decay. We choose the parameters $\alpha_s = 1$, $f_B = 200\,\mathrm{MeV}$, $\bar{\Lambda} = 500\,\mathrm{MeV}$, $\varepsilon = \pm 0.1$ and plot the spectrum for $\lambda = 100\,\mathrm{MeV}$ and $\lambda = 500\,\mathrm{MeV}$. The solid curve is the spectrum without the power correction, which in the present approximation is also the decay spectrum for the neutral $\bar{B}^0$ meson decay. For the smaller value of $\lambda$ the effect can be large, but it is concentrated at $P_+ < 0.5\,\mathrm{GeV}$ as expected. Clearly, this is a rather model-dependent statement. Since the integral of $v_{s,a}(n_-p)$ vanishes, the effect changes sign at some $P_+$. Once a sufficiently large interval is integrated over, the integrated effect is not expected to exceed the estimate (8.8). However, we still expect the four-quark contributions to provide the largest difference between the spectra of charged and neutral $\bar{B}$ meson decays.



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## 9. Conclusions

Using the framework of soft-collinear effective theory, we have investigated the $1/m_b$ corrections to inclusive heavy-to-light transitions in the endpoint region, where the heavy quark expansion in local operators breaks down. We find that SCET factorizes short- and long-distance effects also at sub-leading power. The hadronic tensor, from which all decay spectra are derived, is represented as convolutions of hard coefficient functions, perturbative jet-functions and non-perturbative shape-functions. However, the structure is significantly more complicated than at leading power, because the factorization formula involves multiple convolutions and many shape-functions. We have given the general form of the effective currents and shape-functions that can appear at order $1/m_b$.

We then computed the hadronic tensor to order $1/m_b$ in the tree approximation. In this approximation the power corrections can be parameterized by a few "effective shape-functions" of a single variable $P_+$ only. Compared to earlier results, we find a new contribution from four-quark operators. The other contributions agree with previous results for some differential distributions, but disagree with others. We estimated the numerical effect of the four-quark shape-functions. The effect can lead to a distortion of the $P_+$ spectrum of $B^- \to X_u \ell \bar{\nu}$ relative to the one of $\bar{B}^0 \to X_u \ell \bar{\nu}$. In our model the distortion can be significant at small $P_+$ and could be of order a few percent when integrated over an interval of order $\Lambda_{\rm QCD}$. (The effect vanishes when integrated over all $P_+$.) Our results suggest that it should suffice to include the $1/m_b$ corrections in the tree approximation in the analysis of measurements. This is fortunate, since a prohibitively large number of unknown shape-functions is expected to appear once radiative corrections are included.

While this paper was being written, two articles [26, 27] appeared that also address $1/m_b$ corrections to $\bar{B}$ decay distributions in the soft-collinear effective theory framework. The paper by Lee and Stewart [26] is similar in scope to the present work. Since they use a different representation of SCET, we find it difficult to compare the results in detail, but we can nonetheless make several comments. The structure of factorization as given by our equation (4.13) agrees with their equation (119) on the point that the most general factorization formula contains convolutions over additional tri- and tetra-local shape-functions whose jet-functions vanish at tree-level. However, it appears to us that their basis of SCET currents at relative order $\lambda^2$ cannot be complete because it does not include four-body operators, which implies that the set of power-suppressed jet-functions quoted in their equation (119) is also incomplete. On the other hand, Lee and Stewart observe that $q_s$ in $\mathcal{L}^{(1)}_{\xi q}$ can be replaced by $\not{n}_+\not{n}_- q_s/4$, which implies a simplification of the general decomposition of the four-quark matrix element (4.12). We find that only 6 out of the 16 invariant functions defined in (4.12) can appear to any order in the perturbative expansion. (Two of the 8 functions defined in [26] can be eliminated.) We differ from [26] on the phenomenological implications of our results. On the basis of power counting it is estimated in [26] that the effect of the four-quark contributions is up to 180% in large disagreement with our estimate of 5% based on a simple model. The discrepancy can be attributed to the absence of the colour suppression factor $\varepsilon$ in their estimate and numerical factors. Bosch, Neubert, and Paz [27] restrict their analysis to the tree approximation, but use the position space SCET formalism as we do. Our section 6 has significant overlap with their paper and the tree level results are in complete agreement (as is most evident by comparing our (6.8) to their equation (28)). Our (effective) shape-functions can be obtained from theirs by making the replacements

$$\begin{aligned} S(\omega) &\to S(-\omega) \\ s(\omega) &\to \frac{s_{\rm kin}(-\omega) + C_{\rm mag}(m_b/\mu)s_{\rm mag}(-\omega)}{2} \\ t(\omega) &\to t(-\omega) \\ \tilde{u}(\omega) &\to -u_s(-\omega) - 2v_s(-\omega) \\ \tilde{v}(\omega) &\to u_a(-\omega) - 2v_a(-\omega)\,. \end{aligned} \tag{9.1}$$

These authors perform a more extensive phenomenological study of decay spectra, but they do not consider the four-quark contributions explicitly. Rather, they absorb $v_{s,a}$ into a redefinition of $u_{s,a}$ (their $\tilde{u}, \tilde{v}$). While this is technically possible, it is also misleading, because it hides the fact that the four-quark contributions are different for charged and neutral $\bar{B}$ meson decay. In a recent paper [28] Neubert estimated the numerical effect of the four-quark shape-functions in various models, including the naive vacuum saturation ansatz ($f(k_+) = \delta(k_+)$). When integrated over a sufficiently large region in $P_+$, his result is in qualitative agreement with ours.

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